A • GUIMIER

About mathematical foundation

of the

special relativity

Abstract :
We give an axiomatic way to present the special relativity .

Contents :

(1) Context

This text is a enhanced partial translation of :
https ://archive • org/details/matricesdelorentz/mode/2 up

(2) Basic principles :

(α) We consider a point (or observer) $\boldsymbol{O}$, in an affine space $\boldsymbol{E}$, which models our physical spatial space, of direction $\vec{E}$, vector space in **3** - dimensional Euclidean isomorphic to $\mathbb{R}^3$ • We associate $\boldsymbol{O}$ with an orthogonal coordinate system $\boldsymbol{R}(\boldsymbol{O}, \boldsymbol{x}, \boldsymbol{y}, z)$ with a base $\boldsymbol{B}\left(\boldsymbol{O}, \vec{i}, \vec{j}, \vec{k}\right)$ with its natural Euclidean structure.
We provide $\boldsymbol{O}$ with a clock which measures time $\boldsymbol{t}$.
We suppose that at each point, fixed with respect to $\boldsymbol{O}$, of the coordinate system $\boldsymbol{R}$ is associated with a clock synchronized with that of $\boldsymbol{O}$ which measures the same time $\boldsymbol{t}$.
Synchronizing Clocks allows to have a variable $\boldsymbol{t}$ independent.
We assume that $\boldsymbol{R}$ is Galilean that is to say if a moving point , left to itself, on which acts no force, continues its trajectory in straight line, at one uniform speed • This hypothesis implies the existence of one only time, except for a change of origin or a change of unit.
We can thus construct a vector space with **4** dimensions and a frame $\mathscr{R}\left(\boldsymbol{O}_{t=0}, \boldsymbol{t}, \boldsymbol{x}, \boldsymbol{y}, z\right)$, $\boldsymbol{O}_{t=0}$ representing the point $\boldsymbol{O}$ at time $\boldsymbol{t=0}$ in this space, associated with a base $\mathscr{B}\left(\boldsymbol{O}_{t=0}, c\overline{\tau}, \overline{i}, \overline{j}, \overline{k}\right)$ orthonormal for the bilinear symmetric form form , $\phi(\boldsymbol{t}, \boldsymbol{x}, \boldsymbol{y}, z) = c^2t^2 - x^2 - y^2 - z^2$ (**c** being the speed of light)) .

(β) We consider another point (or observer) $\boldsymbol{O'}$ having a uniform speed $\vec{V}$ relative **to** O and measured by $\boldsymbol{O}$ with which is also associated an orthorormal coordinate system $\boldsymbol{R'}(\boldsymbol{O'}, \boldsymbol{x'}, \boldsymbol{y'}, z')$ associated with a base $\boldsymbol{B'}\left(\boldsymbol{O'}, \vec{i'}, \vec{j'}, \vec{k'}\right)$ with its natural Euclidean structure.We provide $\boldsymbol{O}$ with a clock which measures time $\boldsymbol{t}$.
We suppose that at each point, fixed with respect to $\boldsymbol{O'}$, of the coordinate system $\boldsymbol{R'}$ is associated with a clock synchronized with that of $\boldsymbol{O'}$ which measures the same time t' .
Synchronizing Clocks allows to have a variable $\boldsymbol{t'}$ independent.
We assume that $\boldsymbol{R'}$ is Galilean.
Also we can thus construct a vector space with **4** dimensions and a frame $\mathscr{R}'\left(\boldsymbol{O'}_{t=0}, \boldsymbol{t'}, \boldsymbol{x'}, \boldsymbol{y'}, z'\right)$, $\boldsymbol{O'}_{t'=0}$ representing the point $\boldsymbol{O'}$ at time $\boldsymbol{t'=0}$ in this space, associated with a base $\mathscr{B}\left(\boldsymbol{O'}_{t'=0}, c\overline{\tau'}, \overline{i'}, \overline{j'}, \overline{k'}\right)$ orthonormal for the bilinear symmetric form form , $\phi'(\boldsymbol{t'}, \boldsymbol{x'}, \boldsymbol{y'}, z') = c^2t'^2 - x'^2 - y'^2 - z'^2$ (**c** being the speed of light)) .

We will assume that the **2** observers pass through the same point of $\boldsymbol{E}$ during their journey

and at this time ,
O and O ' set their clock to 0 We will assume that the 2 observers pass through the same point of E during their journey and at this time, O and O 'set their clock to 0 $(t = t' = 0)$.
This nonessential hypothesis simplifies the calculations and we will talk about the bases $\mathscr{B}\left(O , c\overline{\tau} , \overline{i} , \overline{j} , \overline{k}\right)$
and $\mathscr{B}'\left(O , c\overline{\tau'} , \overline{i'}, \overline{j'}, \overline{k'}\right)$ *by setting* $O = O_{t=0} = O'_{t'=0}$.
Otherwise we can consider a third observer O''*, having the same uniform speed .* $\vec{V}$ *relative to O , but whose trajectory, a straight line parallel to that of O', intersects that of O.*
The spatio − temporal units being defined by the physical laws which we will suppose to be the same in the 2 frames , we will choose the same units in the 2 frames.
(γ) *We assume that the photons move in a straight line at speed* **c**, *independently of the considered reference frame. We also assume that* **c** *is the maximum possible speed.*
This implies that for a photon emitted from O at time $t = 0$*, that is to say also from O' at time* $t' = 0$ *, its coordinates in* $\mathscr{B}$ *and* $\mathscr{B}'$ **will** *check simultaneously :*
$c^2t^2 - x^2 - y^2 - z^2 = 0 \Leftrightarrow c^2t'^2 - x'^2 - y'^2 - z'^2 = 0$ *(conservation of the cone of light).*

(3) Relative velocity of 2 frames :
In classical mechanics, if we consider 2 observers O and O' in uniform relative motion to the other we can write that $\overrightarrow{V_{O' \mid O}} = -\overrightarrow{V_{O \mid O'}}$ *for these 2 observers :*
the time is absolute as well as the distance $\left\| \overrightarrow{OO'} \right\|$.
In special relativity , the laws of physics are the same in the 2 frames in uniform relative motion to the other , that is to say that same objects placed under the same conditions will produce the same effects :
Measuring velocity of O' relative to O and measuring the velocity of O relative to O' will give the same result as long as these 2 velocities have the same norm .
As the 2 measured times t and t' are different likewise for spatial coordinates,
Now remembering that at this stage of the study , we only know that the transformation is linear and that the velocity of light is invariant, we will justify in an elementary way that the relative velocity of the 2 frames in uniform translation has the same norm, measured in one or the other frame and vectorially opposite.
Let be two spatial frames $\mathscr{R}$ *and* $\mathscr{R}'$ *in uniform translation. Let assume that their origins O and O' coincide only once at during their relative movement at a point in the spatial space and at this point the clocks of the two spatial frames are set to 0 :* $t = t' = 0$.
We recall that the time associated with any fixed point by relation to O in $\mathscr{R}$ *is synchronized with O .*
Similarly for the time associated with any fixed point by relation to O' in $\mathscr{R}'$ *is synchronized with O' .*
We can thus define the uniform velocity of a point P(t) with respect to O in $\mathscr{R}$ *by:*

$$\left(\vec{\mathcal{V}}_{P \mid O}\right)_{\mathscr{R}} = \frac{\overrightarrow{OP(t_1)} - \overrightarrow{OP(t_0)}}{t_1 - t_0}$$

. Similarly in $\mathscr{R}'$.
Knowing that $\overrightarrow{OO'}(0) = \vec{0}$*, we have in* $\mathscr{R}$ *:*

$$\left(\vec{\mathcal{V}}_{O' \mid O}\right)_{\mathscr{R}} = \frac{\overrightarrow{OO'(t)}}{t} = -\frac{\overrightarrow{O'O(t)}}{t} = -\left(\vec{\mathcal{V}}_{O \mid O'}\right)_{\mathscr{R}} ,$$

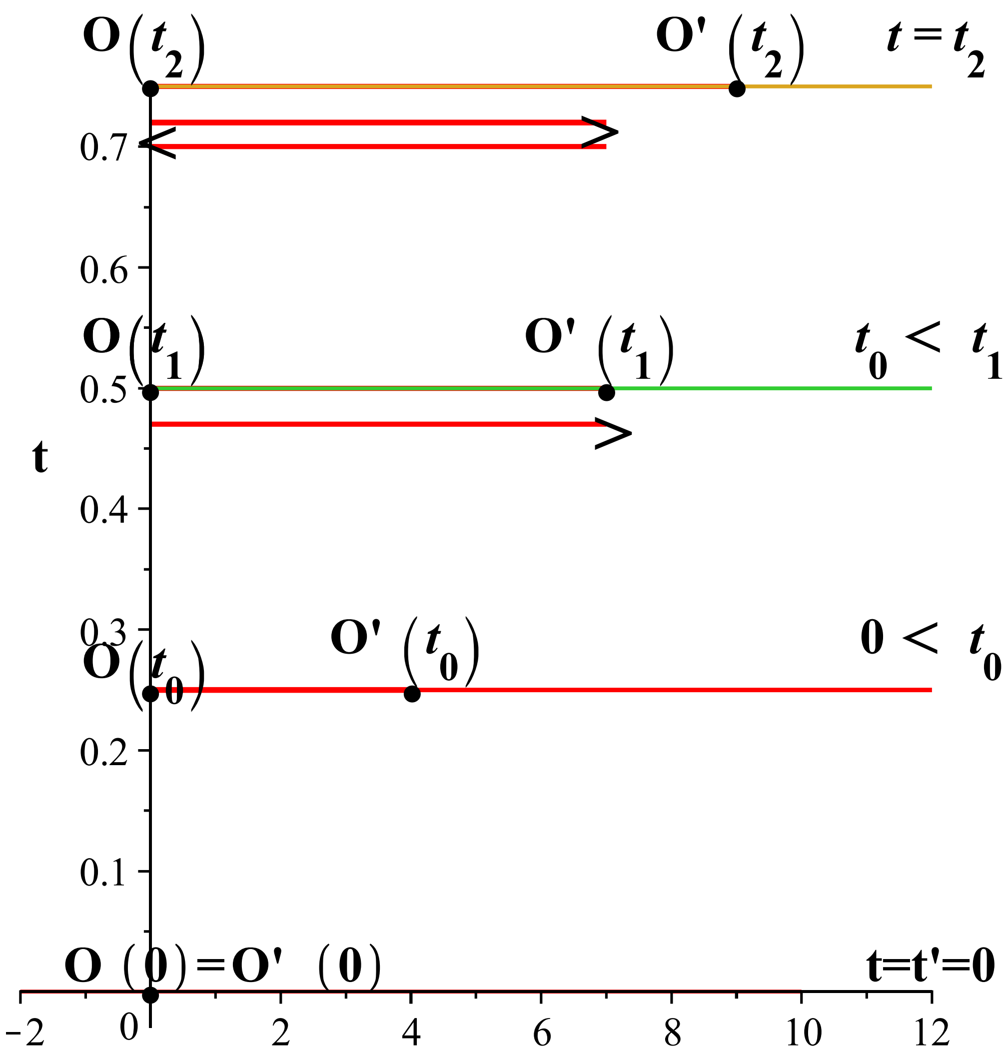
Similarly in $\mathscr{R}'$ *knowing that* $\overrightarrow{O'O(0)} = \vec{0}$ *:*

$$\left(\vec{\mathcal{V}}_{O\,|\,O'}\right)_{\mathcal{R}'} = \frac{\overrightarrow{O'O(t')}}{t'} = -\frac{\overrightarrow{OO'(t')}}{t'} = -\left(\vec{\mathcal{V}}_{O'\,|\,O}\right)_{\mathcal{R}'} \quad .$$

Is true that : $\left(\vec{\mathcal{V}}_{O'\,|\,O}\right)_{\mathcal{R}} = -\left(\vec{\mathcal{V}}_{O\,|\,O'}\right)_{\mathcal{R}'}$ *?*

We already know that these **2** *velocities are parallel to* $\overrightarrow{OO'}$*, constant and in opposite directions.*
Let us evaluate their respective norm Let $\vec{\mathcal{V}}$ *be the vvelocity of* **O '***relative to O in* $\mathcal{R}$*.*
How to measure $\|\vec{\mathcal{V}}\|$ *? For this we are going to carry out a simple experiment measured from* **O** *and* **O'***. We will note* $\mathcal{R}_4$ *and* $\mathcal{R}_4'$ *the frames in the space in* **4** *dimensions, associated with* $\mathcal{R}$ *and* $\mathcal{R}'$ *and at their respective clock .*
As the transformation which makes it possible to pass from $\mathcal{R}_4$ *to* $\mathcal{R}_4'$ *is linear , it can be represented by a matrix* **M**, $\mathcal{R}_4$ *and* $\mathcal{R}_4$ *being provided with adequate orthonormal bases* $\mathcal{B}$ *and* $\mathcal{B}'$ *.*

Experiment :
At time $t = t' = 0$, **O** *and* **O'** *coincide.*
At time $t = t_0 > 0$ *we send a light ray from* **O** *to* **O'** *which returns it to* **O** *at time* $t = t_1$,
and reaches **O** *at time* $t = t_2$.
We denote by $O'(t)$ *and* $O(t)$ *the position of* **O'** *and* **O** *at time* **t in** $\mathcal{R}$*.*
We assume that the axis $\overrightarrow{Ox}$ *is parallel to* $\overrightarrow{OO'}$ *:*



As the movement is uniform rectilinear along the line defined ***by O*** *and* $\vec{V}$*:*
we therefore have $\overline{OO'(t_0)} = \|\vec{V}\|.t_0$, *similarly* $\overline{O'(t_0)O'(t_1)} = \|\vec{V}\|.(t_1 - t_0)$.

The duration being the same on the outward and return journey $t_1 = t_0 + \frac{(t_2 - t_0)}{2} = \frac{(t_2 + t_0)}{2}$.

If ***c*** *is the speed of the light :*

$$(t_2 - t_0)c = 2\|\vec{V}\|\ t_0 + 2\|\vec{V}\|\ (t_1 - t_0)$$

therefore : $(t_2 - t_0)c = 2\|\vec{V}\|\ t_1$ *and therefore :* $\|\vec{V}\| = \frac{(t_2 - t_0)}{2\,t_1}c = \frac{(t_2 - t_0)}{t_2 + t_0}c$.

From the observer's point of view on ***O '****, he sees a ray starting from* ***O*** *at time* $t' = t_0'$,

and $\overline{O'O(t_0')} = \|\vec{V'}\|.t_0'$,.

the ray will arrive at time t_1', *then it will depart towards* ***O*** *which it will touch at time* t_2'.

The ray will therefore have traveled the distance $c(t_2' - t_0')$.

In the direction $\overrightarrow{OO'}$ *the distance traveled is :* $\overline{O'O(t_0')} = \|\vec{V'}\|.t_0'$ *and in the return direction :*

$\|\vec{V'}\|.t_0' + \|\vec{V'}\|.(t_1' - t_0') + \|\vec{V'}\|.(t_2' - t_1') = \|\vec{V'}\|\ t_2'$ *and therefore :*

$(t_2'-t_0')c = \|\vec{\mathcal{V}}'\|\ t_1' + \|\vec{\mathcal{V}}'\|\ t_2'$ *and* $\|\vec{\mathcal{V}}'\| = \dfrac{(t_2'-t'_0)}{t_2'+t_0'}c$.

Let $M = (m_{i,j})$ *be the transformation matrix from* $\mathcal{R}_4'$ *to* $\mathcal{R}_4$

O *has for coordinates* $(ct, 0, 0, 0)$ *in* $\mathcal{R}_4$,

O *will have for coordinates* $(m_{1,1}.ct, m_{2,1}.ct, m_{3,1}.ct, m_{4,1}.ct)$ *dans* $\mathcal{R}_4'$,

so $ct' = m_{1,1}.ct$ *and therefore* $\|\vec{\mathcal{V}}\| = \dfrac{(t'_2 - t'_0)}{t'_2 + t'_0}.c = \dfrac{(t_2 - t_0)}{t_2 + t_0} \cdot c$.

has the same value in $\mathcal{R}'$ *and* $\mathcal{R}$.

The experiment can be seen by the **2** *observers* O *and* O' *as an experiment to measure the relative speed between* O *and* O' *we have :* $\|\vec{\mathcal{V}}_{O'|O}\|_{\mathcal{R}} = \|\vec{\mathcal{V}}_{O|O'}\|_{\mathcal{R}'}$.

We can therefore speak of the relative speed of **2** *frames in uniform translation* $\mathcal{R}$ *and* $\mathcal{R}'$ *with speed* $\vec{\mathcal{V}}$ *et* $-\vec{\mathcal{V}}$ *and with common module* $\|\vec{\mathcal{V}}\|$.

Note : *(See : N.M.J. Woodhouse .* " Special Relativity " *• Springer 2002)*
If we consider an observer on **O** *who observes a clock on* **O'** *which moves away from* **O**, *with a uniform speed and an observer on* **O'** *who observes an identical clock on* **O** *which moves away from* **O'** *we are in a completely symmetrical situation since we have the same speed in module in the two cases The physical laws being the same in the* **2** *Galilean frames in uniform translation the coefficient of expansion of the` durations will be the same in the* **2** *measurements made in each of the* **2** *frames.*
So if we denote $N = (n_{i,j})$ *the inverse matrix of* M, *knowing that* $t' = m_{1,1}.t$ *et* $t = n_{1,1}.t'$,
by *what precedes we will have* $m_{1,1} = n_{1,1}$.
We will then denote γ *this common value .*

To go further we need :

Lemma : *If* $M = (m_{i,j})$ *is symmetric matrix then :* $({}^tXMX = 0, \forall X \in \mathbb{R}^n) \Leftrightarrow M = 0$ **.**

Proof : As M *is symmetric* M *can be diagonalized :* $M = {}^t\Omega D \Omega$ *where* Ω *is orthogonal and* D *diagonal then* ${}^tXMX = {}^tX\,{}^t\Omega D \Omega X = 0$ **.**

If D *is such that* $d_{ij} = 0$ *for* $i \neq j$, *and* $Y = \Omega X$ *then* $\sum_{i=1}^{n} d_{ii} y_i^2 = 0$, $\forall Y \in \mathbb{R}^n$,

for $Y = e_i$, *with* $(e_i)_{i=1,\ldots,n}$ *the canonical basis of* $\mathbb{R}^n$, *we have*
$d_{ii} \cdot 1 = 0 \Rightarrow d_{ii} = 0$ • *We can point out that the result is false*

if M *is not symmetric : take* $M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$: ${}^tXMX = 0\ \forall X \in \mathbb{R}^n$ •

But we have the result $({}^tXMY, \forall X \in \mathbb{R}^n, \forall Y \in \mathbb{R}^n) \Leftrightarrow M = 0$,
since $e_i \cdot M \cdot e_j = m_{i,j}$ • *From this we can deduce again the lemma :*
If M *is symmetric and if we have* ${}^tXMX = 0, \forall X \in \mathbb{R}^n$ (1) *then* $\forall Y \in \mathbb{R}^n$ ${}^tYMY = 0$ (2) ,
and therefore ${}^t(X+Y)M(X+Y) = 0$ (3) ,
so ${}^tXMX + {}^tXMY + {}^tYMX + {}^tYMY = 0$ *and* ${}^tXMY + {}^tYMX = 0$.
M *being symmtreric and antisymmetric* $M = 0$.

Corollary : $^{t}(MX)G(MX)\ ,\forall X\in\mathbb{R}^{n}\ \Leftrightarrow {}^{t}MGM=G.$

Proof :

$$^{t}(MX)G(MX)={}^{t}X\,{}^{t}MGMX={}^{t}XGX,\ \forall X\in\mathbb{R}^{n}\ \Leftrightarrow {}^{t}X\left({}^{t}MGM-G\right)X=0,\ \forall X\in\mathbb{R}^{n},$$

$$\Leftrightarrow {}^{t}MGM=G.$$

Lemma : (*See : N.M.J. Woodhouse*. " Special Relativity " •*Springer 2002*)

Let $A\in M_4(\mathbb{R})$ *be a symmetrc matrix and* $G=\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}$.

Let's assume $\forall X\ :\ {}^{t}XGX=0\Rightarrow {}^{t}XAX=0$,

then $\exists\,\alpha\in\mathbb{R}$ *such as* $A=\alpha G$.

Proof **:**

If A *is symmetric we can write as :*

$A=\begin{bmatrix}\alpha&{}^{t}a\\a&S\end{bmatrix}$ *where* $\alpha\in\mathbb{R},\ a\in M_{3,1}(\mathbb{R})\ ,S\in M_3(\mathbb{R})$ S *symmetric* .

We compute for ${}^{t}X=[u,{}^{t}Y],\ Y\in M_{3,1}(\mathbb{R})\ ,r\in\mathbb{R}$**:**

$${}^{t}XAX=[r,{}^{t}Y]\begin{bmatrix}\alpha&{}^{t}a\\a&S\end{bmatrix}\begin{bmatrix}r\\Y\end{bmatrix}=\alpha r^{2}+r\,{}^{t}Ya+r\,{}^{t}aY+{}^{t}YSY\ \text{ then :}$$

$${}^{t}XMX=\alpha t^{2}+2\,r\,{}^{t}aY+{}^{t}YSY\ .(1)$$

We note ${}^{t}U=[u,v,w]$ *with* $u^{2}+v^{2}+w^{2}=1$ *et* ${}^{t}X=[1,U]$ (2)

In this case ${}^{t}XGX=0$ *and then* ${}^{t}XAX=0$,

and (1) *can be written with* $r=1\ ,\ Y=U$.

$\forall U$ *defined by* (2) **:** ${}^{t}XAX=\alpha+2\,{}^{t}aU+{}^{t}USU=0\,.(3)$

If U *verifies* (3) $-U$ *verifies* (2) *and* (3) *then:*

$$\alpha+2\,{}^{t}a(-U)+{}^{t}(-U)S(-U)=0\ \ (4)\ .$$

By adding (3) *and* (4) *it follows :*

$\forall U$ *defined by* (2) : $\alpha+{}^{t}USU=0\Leftrightarrow {}^{t}U\left(\alpha Id_{3}+S\right)U=0.$

If ${}^{t}U\left(\alpha Id_{3}+S\right)U=0$ $\forall U$ *defined by* (2)

${}^{t}U\left(\alpha Id_{3}+S\right)U=0$ *is also true for* $\forall U\in\mathbb{R}^{3}$**.**

According to the previous lemma : $S=-\alpha\cdot Id_{3}$.

By substracting (3) *et* (4) *we have :* $\forall U\ \ {}^{t}aU=0\ \Rightarrow a=0$.

Then :

$$A=\alpha\cdot\begin{bmatrix}\alpha&0&0&0\\0&-\alpha&0&0\\0&0&-\alpha&0\\0&0&0&-\alpha\end{bmatrix}=\alpha G.$$

This result is a special case of a more general result :
Let be ϕ and ϕ' be two symmetric bilinear forms such the isotropic cone $\mathcal{C}(\phi) \neq \{0\}$
then $\exists \lambda \neq 0 \in \mathbb{R}\ \phi = \lambda \phi' \Leftrightarrow \mathcal{C}(\phi) = \mathcal{C}(\phi')$.
(*cf* : *R.Goblot* ."Algébre linéaire "*Masson 1995* .)

Now we can prove :
Theorem : *Let* $M = (m_{i,j})$ *be the transformation matrix from* $\mathcal{R}_4'$ *to* $\mathcal{R}_4$
Let's assume $\forall X\ :\ {}^tXGX = 0 \Rightarrow {}^tX{}^tMGMX = 0\ \ (1)$,

$$(M)_{1,1} = (M^{-1})_{1,1}\ \ (2) \quad \text{then}$$

$${}^tMGM = G$$

Proof : (*cf* : *N • M • J • Woodhouse* • " Special Relativity " •*Springer 2002*)
From (*1*) *and the previous lemma , since* ${}^tMGM$ *is symmetric ,*
${}^tMGM = \alpha G$ *for some* $\alpha \in \mathbb{R}$, $\alpha \neq 0$ *since* M *is non singular.*
Hence $({}^tMGM)^{-1} = (\alpha G)^{-1} = \alpha^{-1} G$ *and*

$$M^{-1}G = \alpha^{-1} G\ {}^tM \Rightarrow M^{-1} = \alpha^{-1} G\ {}^tMG .$$

We have $(M)_{1,1} = ({}^tM)_{1,1} = (M^{-1})_{1,1} = (\alpha^{-1} G\ {}^tMG)_{1,1}$ *by* (*2*)
and

$$G\ {}^tMG = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} . \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Hence $(G\ {}^tMG)_{1,1} = m_{1,1}$ *and* $\alpha = 1$.

Note : (*1*) *is a consequence of the invariance of the light cone .*
For (*2*) *see the previous note .*

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(*4*) ***Lorentz matrix study :***

We start by giving the general properties of Lorentz matrices then
we give an estimation of each term of these matrices .
(*We note* ${}^tX$ *and* ${}^tM$ *the transpose of the column vector* X *and the matrix* M) .
Introduction :

Let $\Phi(X) = x_1^2 - \sum_{i=2}^{n} x_i^2$ *with* $({}^tX = (x_1, ..., x_n))$ *be the quadratic Lorentz*
form and let G *be the matrice* $\begin{bmatrix} 1 & {}^t0 \\ 0 & -Id_{\mathbb{R}^{n-1}} \end{bmatrix}$ *where* 0 *is the zero - column of* $\mathbb{R}^{n-1}$:

$\Phi(X) = {}^tXGX\ \ \forall X \in \mathbb{R}^n$.
We seek the matrices M *such*

$$\Phi(MX) = \Phi(X) \Leftrightarrow {}^t(MX)G(MX) = {}^tX\,{}^tMGMX = {}^tXGX,\ \forall X \in \mathbb{R}^n.$$

Lemma : If $M=(m_{i,j})$ is symmetric then : $({}^tXMX=0, \forall X\in\mathbb{R}^n) \Leftrightarrow M=0$.

Proof : As M is symmetric M can be diagonalized : $M={}^t\Omega D\Omega$ where Ω is orthogonal and D diagonal then ${}^tXMX={}^tX\,{}^t\Omega D\Omega X=0$.

If D is such that $d_{ij}=0$ for $i\neq j$, and $Y=\Omega X$ then $\sum_{i=1}^{n} d_{ii}y_i^2=0$, $\forall Y\in\mathbb{R}^n$,

for $Y=e_i$, with $(e_i)_{i=1,\dots,n}$ the canonical basis of $\mathbb{R}^n$, we have

$d_{ii}\cdot 1=0 \Rightarrow d_{ii}=0$. We can point out that the result is false

if M is not symmetric : take $M=\begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix}$: ${}^tXMX=0\ \forall X\in\mathbb{R}^n$.

But we have the result $({}^tXMY, \forall X\in\mathbb{R}^n, \forall Y\in\mathbb{R}^n) \Leftrightarrow M=0$,
since $e_i\cdot M\cdot e_j=m_{i,j}$. From this we can deduce again the lemma :

If M is symmetric and if we have ${}^tXMX=0, \forall X\in\mathbb{R}^n$ (1) then $\forall Y\in\mathbb{R}^n$ ${}^tYMY=0$ (2) ,
and therefore ${}^t(X+Y)M(X+Y)=0$ (3) ,
so ${}^tXMX+{}^tXMY+{}^tYMX+{}^tYMY=0$ and ${}^tXMY+{}^tYMX=0$.
M being symmtreric and antisymmetric $M=0$.

<u>Corollary</u> : ${}^t(MX)G(MX)$, $\forall X\in\mathbb{R}^n \Leftrightarrow {}^tMGM=G$.

Proof :

${}^t(MX)G(MX)={}^tX\,{}^tMGMX={}^tXGX, \forall X\in\mathbb{R}^n \Leftrightarrow {}^tX({}^tMGM-G)X=0, \forall X\in\mathbb{R}^n$,

$\Leftrightarrow {}^tMGM=G$.

<u>Definition</u> :

If M is such ${}^tMGM=G$, M is called a Lorentz matrix .

<u>Theorem 1 :</u>

The Lorentz matrices form a subgroup L of $GL_n(\mathbb{R})$, group of the invertible matrices of dimension n.

Proof : (1) As $Id_{\mathbb{R}^n}\in L$, $L\neq\varnothing$ and $\det({}^tMGM)=\det(G)=-1 \Rightarrow \det(M)\neq 0$,

$\Rightarrow \{Id_{\mathbb{R}^n}\}\subseteq L\subseteq GL_n(\mathbb{R})$.

(2) ${}^tMGM=G \Rightarrow G=G^{-1}=M^{-1}G({}^tM)^{-1}=M^{-1}G\,{}^t(M^{-1})$.

(3) If $M\in L$ and $N\in L$, we have ${}^t(MN)G(MN)={}^tN({}^tMGM)N=G$.

From (1), (2) and (3) we can infer the theorem .

Corollary :

$${}^tMGM=G \Rightarrow M^{-1}=G\,{}^tM\,G=\begin{bmatrix} m_{1,1} & -m_{2,1} & -m_{3,1} & -m_{4,1}\\ -m_{1,2} & m_{2,2} & m_{3,2} & m_{4,2}\\ -m_{1,3} & m_{2,3} & m_{3,3} & m_{4,3}\\ -m_{1,4} & m_{2,4} & m_{3,4} & m_{4,4}\end{bmatrix} \text{ if } M=(m_{i,j}).$$

A Lorentz matrix being invertible by (1) , M has a unique polar decomposition :
$M=SO$, S symmetric and definite positive , O orthogonal .
We refer to a classical theorem : (theorem of decomposition)
Any invertible matrix A can be written, in a unique way, as a product : $A=SO=O_1S_1$,

where S, S_1 *are positive definite symmetric matrices ,* O, O_1 *are orthogonal matrices,*

here : $S=\sqrt{{}^tAA}$, $S=\sqrt{A\,{}^tA}$.

(*Cf. F.R.Gantmacher : Theory of matrices. AMS Chelsea Publishing 1959*).

In the case of **L** *we can be more specific .*

For that we need some lemmas **.**

(*Cf. J-M. Souriau.*"Calcul Linéaire" •*PUF 1964 or J •Gabay 2000*)

Lemma 1 :

Let $X \in \mathbb{R}^n$ *be such that* ${}^tXX=1$ •*Let* **N** *the matrix define by :*

$\mathcal{O}_{n-1}$ *being the null matrix of* $\mathbb{M}_{n-1}(\mathbb{R})$,

$$N=\begin{bmatrix} 0 & {}^tX \\ X & \mathcal{O}_{n-1}\end{bmatrix}$$ **.** *Then we have* $\forall \alpha \in \mathbb{R}$:

$$M=exp(\alpha N) = \begin{bmatrix} ch(\alpha) & sh(\alpha)\,{}^tX \\ sh(\alpha)X & \left(Id_{\mathbb{R}^{n-1}}+(ch(\alpha)-1)X\,{}^tX\right)\end{bmatrix} \qquad (1)$$

is a Lorentz matrice of $\mathbb{M}_n(\mathbb{R})$ *such as* $det(M)=1$ *and its eingenvalues are strictly positive .*

In short M is a definite positive symmetric Lorentz matrix .

Proof : We have $N^2=\begin{bmatrix} 1 & 0 \\ 0^{n-1} & X\,{}^tX\end{bmatrix}$ 0^{n-1} *being the zero - column of dimension* $n-1$,

since ${}^tXX=1$ *and* $(X\,{}^tX)(X\,{}^tX) = X({}^tXX)\,{}^tX$ *we have* $N^3=N$.

We can notice that for all square matrix **A** *and for all* $\alpha \in \mathbb{R}$,

$$exp(A) = \sum_{n=0}^{+\infty}\frac{A^n}{n!} \quad , \; sh(\alpha) = \sum_{n=0}^{+\infty}\frac{\alpha^{2n+1}}{(2n+1)!} \; , \; ch(\alpha) = \sum_{n=0}^{+\infty}\frac{\alpha^{2n}}{(2n)!} \, ,$$

therefore

$$exp(\alpha N) = Id_{\mathbb{R}^4} + \alpha N + \frac{\alpha^2 N^2}{2!} + \frac{\alpha^3 N}{3!} + \frac{\alpha^4 N^2}{4!} + ... +$$

$$+ \frac{\alpha^{2p-1}N}{(2p-1)!} + \frac{\alpha^{2p}N^2}{(2p)!} + ...$$

$$= Id_{\mathbb{R}^n} + sh(\alpha)N + (ch(\alpha)-1)N^2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ | & \diagdown & | \\ 0 & 0 & 1\end{bmatrix} + \begin{bmatrix} 0 & sh(\alpha)\,{}^tX \\ sh(\alpha)X & \mathcal{O}_{n-1}\end{bmatrix} + \begin{bmatrix} (ch(\alpha)-1) & 0 \\ 0^{n-1} & (ch(\alpha)-1)X\,{}^tX\end{bmatrix}$$ *which shows* (**1**) .

Now let's show that **M** *is of Lorentz that is to say* ${}^tMGM=G$ **.**

We can remark that **M** *is symmetric and with* $Id=Id_{\mathbb{R}^{n-1}}$

$$\begin{bmatrix} ch(\alpha) & sh(\alpha)\,{}^tX \\ sh(\alpha)X & \left(Id+(ch(\alpha)-1)X\,{}^tX\right)\end{bmatrix}\begin{bmatrix} 1 & {}^t0^{n-1} \\ 0^{n-1} & -Id\end{bmatrix} = \begin{bmatrix} ch(\alpha) & -sh(\alpha)\,{}^tX \\ sh(\alpha)X & -\left(Id+(ch(\alpha)-1)X\,{}^tX\right)\end{bmatrix}$$

and

$$\begin{bmatrix} ch(\alpha) & -sh(\alpha)\,{}^tX \\ sh(\alpha)X & -\left(Id + (ch(\alpha)-1)X{}^tX\right) \end{bmatrix}\begin{bmatrix} ch(\alpha) & sh(\alpha)\,{}^tX \\ (sh(\alpha))X & \left(Id + (ch(\alpha)-1)X{}^tX\right) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Let us estimate **a, b, c, d** .

$a = ch^2(\alpha) - sh^2(\alpha)\,{}^tXX = 1$, *since* ${}^tXX = 1$ **.**

$b = ch(\alpha)\ sh(\alpha)\,{}^tX - sh(\alpha)\,{}^tX\left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha)-1)X{}^tX\right)$

$= ch(\alpha)\ sh(\alpha)\,{}^tX - sh(\alpha)\,{}^tX - sh(\alpha)(ch(\alpha)-1)\,{}^tXX{}^tX$

$= ch(\alpha)\ sh(\alpha)\,{}^tX - sh(\alpha)\,{}^tX - sh(\alpha)ch(\alpha)\,{}^tX + sh(\alpha){}^tX = 0$,

$c = ch(\alpha)\ sh(\alpha)X - \left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha)-1)X{}^tX\right)sh(\alpha)X$

$= ch(\alpha)\ sh(\alpha)X - sh(\alpha)X - sh(\alpha)(ch(\alpha)-1)X = 0$,

$d = sh^2(\alpha)X{}^tX - \left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha)-1)X{}^tX\right)^2$ *but :*

$\left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha)-1)X{}^tX\right)\left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha)-1)X{}^tX\right)$

$= Id_{\mathbb{R}^{n-1}} + (ch(\alpha)-1)X{}^tX + (ch(\alpha)-1)X{}^tX + (ch(\alpha)-1)^2X{}^tXX{}^tX,$

since ${}^tXX = 1$ *then* $X({}^tXX){}^tX = X{}^tX$ (*association of matrix product*) ,

et $(ch(\alpha)-1)^2 = ch^2(\alpha) - 2ch(\alpha) + 1$

$= Id_{\mathbb{R}^{n-1}} + \left(2(ch(\alpha)-1) + ch^2(\alpha) - 2ch(\alpha) + 1\right)X{}^tX$

$= Id_{\mathbb{R}^{n-1}} + \left(ch^2(\alpha) - 1\right)X{}^tX = Id_{\mathbb{R}^{n-1}} + sh^2(\alpha)X{}^tX$

finally :

$sh^2(\alpha)X{}^tX - \left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha)-1)X{}^tX\right)^2 = -Id_{\mathbb{R}^{n-1}}$.

Therefore $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & {}^t0 \\ 0 & -Id_{\mathbb{R}^{n-1}} \end{bmatrix}$

and **M** *is of* **Lorentz** .

Another proof possible is :

$$exp(\alpha N)G = \begin{bmatrix} ch(\alpha) & sh(\alpha)\,{}^tX \\ sh(\alpha)X & \left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha)-1)X{}^tX\right) \end{bmatrix}\begin{bmatrix} 1 & {}^t0 \\ 0 & -Id_{\mathbb{R}^{n-1}} \end{bmatrix}$$

$$= \begin{bmatrix} ch(\alpha) & -sh(\alpha)\,{}^tX \\ sh(\alpha)X & -\left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha)-1)X{}^tX\right) \end{bmatrix},$$

$$Gexp(-\alpha N) = \begin{bmatrix} 1 & {}^t0 \\ 0 & -Id_{\mathbb{R}^{n-1}} \end{bmatrix}\begin{bmatrix} ch(\alpha) & -sh(\alpha)\,{}^tX \\ -sh(\alpha)X & \left(Id_{\mathbb{R}^{n-1}} + (ch(\alpha)-1)X{}^tX\right) \end{bmatrix}$$

$$= \begin{bmatrix} ch(\alpha) & -sh(\alpha)\,{}^tX \\ sh(\alpha)X & -\left(Id_{\mathbb{R}n-1} + (ch(\alpha)-1)X{}^tX\right) \end{bmatrix}$$

therefore $\boldsymbol{exp}(\alpha N)\boldsymbol{G} = \boldsymbol{Gexp}(-\alpha N) \Rightarrow \boldsymbol{exp}(\alpha N)\boldsymbol{Gexp}(\alpha N) = \boldsymbol{G}$

as $\boldsymbol{M} = \boldsymbol{exp}(\alpha N)$ *est symétrique* ${}^t\boldsymbol{MGM} = \boldsymbol{G}$ *et* $\boldsymbol{M}$ *est de* **Lorentz.**

For a third demonstration see : J-M. Souriau."Calcul Linéaire" *•PUF 1964 or J •Gabay 2000*.

Let us now show that $\boldsymbol{det}(\boldsymbol{exp}(\boldsymbol{A})) = \boldsymbol{e}^{Tr(A)}$ *for* $\boldsymbol{A} \in \boldsymbol{M}_n(\mathbb{R})$ *:*

Let us consider $\boldsymbol{A}$ *as an element of* $\boldsymbol{M}_n(\mathbb{C})$ *.* $\boldsymbol{A}$ *is then trigonalisable and can be written :* $\boldsymbol{A} = \boldsymbol{P}^{-1}\boldsymbol{BP}$ *, with* $\boldsymbol{B}$ *upper triangular whose diagonal is composed of the eigenvalues* λ_i *of* $\boldsymbol{A}$.

By relying on the definition of the exponential of a matrix we can write $\boldsymbol{exp}(\boldsymbol{A}) = \boldsymbol{P}^{-1}\boldsymbol{exp}(\boldsymbol{B})\boldsymbol{P}$.

By expanding $\boldsymbol{exp}(\boldsymbol{B})$ *into series and by noticing that* $\boldsymbol{B}^k$ *is upper triangular* $\forall k \geq 1$.

Then $\boldsymbol{exp}(\boldsymbol{B})$ *has for diagonal the* $e^{\lambda_i} > 0$. *Therefore*

$$\boldsymbol{det}(\boldsymbol{exp}(\boldsymbol{A})) = \boldsymbol{det}(\boldsymbol{exp}(\boldsymbol{B})) = \prod_i e^{\lambda_i} = e^{\sum_i \lambda_i} = e^{Tr(A)}.$$

Let's go back to $\boldsymbol{exp}(\alpha N)$ *:*

As αN *is a zero − trace matrix :* $\boldsymbol{det}(\boldsymbol{M}) = \boldsymbol{det}(\boldsymbol{exp}(\alpha N)) = 1$ *and the eingenvalues of* $\boldsymbol{M} = \boldsymbol{exp}(\alpha N)$ *are strictly positive .*

Lemma 2 :

If M *is a matrix of Lorentz whose first column* K_1 *is of the form :* $K_1 = {}^t(\alpha, 0,..., 0)$ *then* M *is of the form :*

$$M = \begin{bmatrix} \varepsilon & {}^tQ \\ Q & \Omega \end{bmatrix} \qquad \varepsilon = \pm 1 , {}^tQ = (0, 0, 0), {}^t\Omega\Omega = Id_{\mathbb{R}^{n-1}} \quad (1).$$

Conversely any matrix of this form is of Lorentz.

Proof :

Let $M = \begin{bmatrix} \alpha & {}^tL \\ 0 & C \end{bmatrix}$ *with* $C \in \mathbb{M}_3(\mathbb{R})$, ${}^tL = (l_1, l_2, l_3)$.

As $({}^tMG)M = G$ *we have :*

$$G = \begin{bmatrix} \alpha & 0 \\ L & -C \end{bmatrix} \begin{bmatrix} \alpha & {}^tL \\ 0 & C \end{bmatrix}$$

$$= \begin{bmatrix} \alpha^2 & \alpha {}^tL \\ \alpha L & L {}^tL - {}^tCC \end{bmatrix}$$

by identification $\alpha = \pm 1$, $L = 0$, ${}^tCC = Id_{\mathbb{R}^{n-1}}$.

Conversely : $\begin{bmatrix} \varepsilon & {}^tQ \\ Q & {}^t\Omega \end{bmatrix} G \begin{bmatrix} \varepsilon & {}^tQ \\ Q & \Omega \end{bmatrix} = G$.

Theorem :

Every matrix M *of Lorentz can be put in the form :*

$$M = exp(\alpha N) \cdot \begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix} \quad \text{with } \alpha \in \mathbb{R}, \varepsilon = \pm 1 ,$$

$N = \begin{bmatrix} 0 & {}^tX \\ X & O \end{bmatrix}$ *, with* $X \in \mathbb{R}^{n-1}$ *such as :* ${}^tXX = 1$, ${}^t\Omega\Omega = Id_{\mathbb{R}^{n-1}}$.

This polar decomposition is unique .

Proof :

Let M *be a matrix of Lorentz* $M = (m_{i,j})$ *and* M_1 *the first column of* M.

If $M_1 = (m_{i,1})$ *then :*

$${}^tM_1GM_1 = m_{1,1}^2 - m_{1,2}^2 - ... - m_{1,n}^2 = G_{1,1} = 1 .$$

We write ${}^tM_1 = (\beta, Y)$ *with* $\beta = m_{1,1}$ *and* ${}^tY = (m_{1,2}, ..., m_{1,n})$.

Therefore $\beta^2 - {}^tYY = 1$ *and* $|\beta| \geq 1$ *. We can write* $\beta = \varepsilon \cdot ch(\alpha)$ *with* $\varepsilon = \pm 1$ *and* $\alpha \in \mathbb{R}$ *. Moreover* ${}^tYY = \beta^2 - 1 = sh^2(\alpha)$.

If $\alpha \neq 0$ *put* $X = \dfrac{Y}{\varepsilon \cdot sh(\alpha)}$ *, we have* ${}^tXX = 1$.

If $\alpha = 0$ on a ${}^tM_1 = (\varepsilon, 0, 0, 0)$.

In all cases ${}^tM_1 = \varepsilon(ch(\alpha), sh(\alpha)X)$ avec ${}^tXX = 1$.

M_1 is the first column of $\varepsilon \cdot exp(\alpha N)$ with $N = \begin{bmatrix} 0 & {}^tX \\ X & O \end{bmatrix}$.

$exp(-\alpha N) \cdot M$ is a matrix of Lorentz since M and $exp(-\alpha N)$ are of Lorentz (**lemma 1**).
Let's estimate the first column K_1 of this product :

$$K_1 = \begin{bmatrix} ch(\alpha) & (sh(-\alpha))\,{}^tX \\ (sh(-\alpha))X & Id_{\mathbb{R}^3} + (ch(\alpha) - 1)X{}^tX \end{bmatrix} \cdot M_1$$

${}^tXX = 1$, $ch^2(\alpha) - sh^2(\alpha) = 1$ and
$\left(Id_{\mathbb{R}^3} + (ch(\alpha) - 1)X{}^tX\right) \cdot X = ch(\alpha) \cdot X$ since $(X{}^tX)X = X({}^tXX)$

We have : ${}^tK_1 = (\varepsilon, 0, .., 0)$ • The lemma 2 implies :

$$M = exp(\alpha N) \cdot \begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix}.$$

αN being a symmetric matrix , it is a diagonalizable matrix : $\alpha N = {}^tPDP$,
with D a diagonal matrix and P an orthogonal matrix :
therefore $exp(\alpha N) = exp({}^tPDP) = {}^tPexp(D)P$ and
$exp(\alpha N)$ is a definite positive matrix .

Moreover $\begin{bmatrix} \varepsilon & 0 \\ 0 & {}^t\Omega \end{bmatrix} \begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix} = Id_{\mathbb{R}^n}$.

We have decompozed M into an product of a definite positive symmetric matrix and an orthogonal one ,by the theorem of decomposition , there is uniqueness .

Note :

(1) If $n = 2$ then $\Omega = \pm 1$ and if $\Omega \cdot \varepsilon = 1$ then M is symmetric .

If $n = 3$ then $\Omega = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$ or $\Omega = \begin{bmatrix} cos(\varphi) & sin(\varphi) \\ sin(\varphi) & -cos(\varphi) \end{bmatrix}$ with $\theta \in \mathbb{R}$, $\varphi \in \mathbb{R}$.

If $n = 4$ then Ω is a spatial rotation if $det(\Omega) = +1$.

If not Ω is a reflection, or a combination of a rotation and a reflection .

(2) Let's consider ${}^tW' = (ct', 0, 0, 0)$ with c a constant , t' a real variable > 0 and let $W = MW'$.

Let 's assume that ${}^tW$ can be written in the form : ${}^tW = (ct, ct\beta_1, ct\beta_2, ct\beta_3)$
with $\beta_1, \beta_2, \beta_3$ real constants , t a real variable.

If $M = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta}) \\ \gamma\vec{\beta} & C \end{bmatrix} \begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix}$ then $t = \varepsilon \cdot \gamma \cdot t'$ and $\varepsilon = \dfrac{t \cdot t'}{|t \cdot t'|}$.

Now, we are going to explicit M in the case of the special relativity :

Lemma 3 :

Let M be a matrix of Lorentz with $n = 4$, $\varepsilon = 1$ and let's consider ${}^tW' = (ct', 0, 0, 0)$
with c a constant , $t' > 0$ a real variable and let $W = MW'$.
Let 's assume that ${}^tW$ can be written in the form :
${}^tW = (ct, ct\beta_1, ct\beta_2, ct\beta_3)$ with $\beta_1, \beta_2, \beta_3$ real constants , t a real variable.

Then if $M = exp(\alpha N) \cdot \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}$, with $N = \begin{bmatrix} 0 & {}^tX \\ X & \mathcal{O}_{n-1} \end{bmatrix}$ and $\mathcal{O}_{n-1}$ the null matrix

then

if we write $\vec{\beta} = {}^t(\beta_1, \beta_2, \beta_3)$ and $\vec{X} = {}^t(X_1, X_2, X_3)$,
we have $exp(\alpha N) = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta}) \\ \gamma\vec{\beta} & C \end{bmatrix}$ and then $M = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta})\Omega \\ \gamma\vec{\beta} & C\Omega \end{bmatrix}$,

$$\text{with} \qquad C = \begin{bmatrix} 1 + \dfrac{\gamma^2}{(1+\gamma)}\beta_1^2 & \dfrac{\gamma^2}{(1+\gamma)}\beta_1\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\beta_1\beta_3 \\ \dfrac{\gamma^2}{(1+\gamma)}\beta_2\beta_1 & 1 + \dfrac{\gamma^2}{(1+\gamma)}\beta_2^2 & \dfrac{\gamma^2}{(1+\gamma)}\beta_2\beta_3 \\ \dfrac{\gamma^2}{(1+\gamma)}\beta_3\beta_1 & \dfrac{\gamma^2}{(1+\gamma)}\beta_3\beta_2 & 1 + \dfrac{\gamma^2}{(1+\gamma)}\beta_3^2 \end{bmatrix} ,$$

$\vec{\beta} = th(\alpha)\vec{X}$ and $\gamma = ch(\alpha)$.

Proof :

First we can point out that $\begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} W' = W'$ donc $W = exp(\alpha N) W'$.

We have to solve :

$$\begin{bmatrix} ch(\alpha) & sh(\alpha)\,{}^tX \\ sh(\alpha)X & \left(Id_{\mathbb{R}^3} + (ch(\alpha)-1)X{}^tX\right) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} t' = \begin{bmatrix} 1 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} t\ .$$

This implies $t\begin{bmatrix} 1 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = t'\begin{bmatrix} ch(\alpha) \\ sh(\alpha)X_1 \\ sh(\alpha)X_2 \\ sh(\alpha)X_3 \end{bmatrix}$ *therefore* $t = ch(\alpha)t'$ *and*

$ch(\alpha)\begin{bmatrix} 1 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} ch(\alpha) \\ sh(\alpha)X_1 \\ sh(\alpha)X_2 \\ sh(\alpha)X_3 \end{bmatrix}$ *for* $t' \neq 0 \Rightarrow ch(\alpha)\vec{\beta} = sh(\alpha)\vec{X}$

therefore $\vec{\beta} = th(\alpha)\vec{X} \Rightarrow \vec{\beta}^2 = th^2(\alpha)$ *since* $\vec{X}^2 = 1$.

We put $\beta = \sqrt{\vec{\beta}^2}$ *and* $\gamma = \dfrac{1}{\sqrt{1-\vec{\beta}^2}}$.

As $1 - th^2(\alpha) = \dfrac{1}{ch^2(\alpha)} \Rightarrow ch^2(\alpha) = \dfrac{1}{1 - th^2(\alpha)} = \dfrac{1}{1-\vec{\beta}^2} = \gamma^2$,

As $ch(\alpha) \geq 1$ *we have* $\gamma = ch(\alpha)$;

As $sh^2(\alpha) = ch^2(\alpha) - 1 = \gamma^2 - 1 = \dfrac{1}{1-\vec{\beta}^2} - 1 = \dfrac{\beta^2}{1-\beta^2} = \gamma^2\beta^2$.

Let's sum up $\gamma = ch(\alpha)$, $\gamma^2\beta^2 = sh^2(\alpha)$, $\beta^2 = th^2(\alpha)$.
On the other hand :

$$\gamma^2\beta^2 = \gamma^2 - 1 = (\gamma+1)(\gamma-1) \Rightarrow \frac{\gamma^2\beta^2}{(1+\gamma)} = (\gamma - 1) = ch(\alpha) - 1$$

and $X_iX_j = \dfrac{(\beta_i\beta_j)}{th^2(\alpha)} = \dfrac{(\beta_i\beta_j)}{\beta^2}$ *therefore :*

$(ch(\alpha) - 1)X_iX_j = \dfrac{\gamma^2\beta^2}{(1+\gamma)}\dfrac{(\beta_i\beta_j)}{\beta^2} = \dfrac{\gamma^2}{(1+\gamma)}\beta_i\beta_j$, *that implies:*

$$Id_{\mathbb{R}^3} + (ch(\alpha) - 1)X^tX = \begin{bmatrix} 1 + \frac{\gamma^2}{(1+\gamma)}\beta_1^2 & \frac{\gamma^2}{(1+\gamma)}\beta_1\beta_2 & \frac{\gamma^2}{(1+\gamma)}\beta_1\beta_3 \\ \frac{\gamma^2}{(1+\gamma)}\beta_2\beta_1 & 1 + \frac{\gamma^2}{(1+\gamma)}\beta_2^2 & \frac{\gamma^2}{(1+\gamma)}\beta_2\beta_3 \\ \frac{\gamma^2}{(1+\gamma)}\beta_3\beta_1 & \frac{\gamma^2}{(1+\gamma)}\beta_3\beta_2 & 1 + \frac{\gamma^2}{(1+\gamma)}\beta_3^2 \end{bmatrix}$$

$$= Id_{\mathbb{R}^3} + \frac{\gamma^2 \vec{\beta}\, {}^t\overleftrightarrow{\beta}}{(1+\gamma)} = C$$

As $sh(\alpha)X_i = \dfrac{sh(\alpha)\beta_i}{th(\alpha)} = ch(\alpha)\beta_i = \gamma\beta_i$ finally we have

$$exp(\alpha N) = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta}) \\ \gamma\vec{\beta} & C \end{bmatrix}.$$

Then $M = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta}) \\ \gamma\vec{\beta} & C \end{bmatrix} . \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta})\Omega \\ \gamma\vec{\beta} & C\Omega \end{bmatrix}.$

Corollary : As $M^{-1} = G\,{}^tM\,G = \begin{bmatrix} m_{1,1} & -m_{2,1} & -m_{3,1} & -m_{4,1} \\ -m_{1,2} & m_{2,2} & m_{3,2} & m_{4,2} \\ -m_{1,3} & m_{2,3} & m_{3,3} & m_{4,3} \\ -m_{1,4} & m_{2,4} & m_{3,4} & m_{4,4} \end{bmatrix}$ if $M = (m_{i,j})$,

$M^{-1} = \begin{bmatrix} \gamma & -{}^t(\gamma\vec{\beta}) \\ -\gamma\,{}^t\Omega\vec{\beta} & {}^t\Omega C \end{bmatrix}$, M^{-1} is of Lorentz $\Rightarrow M^{-1} = \begin{bmatrix} \gamma & -{}^t(\gamma\vec{\beta})\Omega' \\ -\gamma\vec{\beta} & C\Omega' \end{bmatrix}$

then ${}^t\Omega\vec{\beta} = \vec{\beta} \Rightarrow \overleftrightarrow{\beta}\Omega = \overleftrightarrow{\beta}$ and $M = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta}) \\ \gamma\vec{\beta} & C\Omega \end{bmatrix}$.

Note : If $C\Omega = [u_1, u_2, u_3]$ then $u_1^2 + u_2^2 + u_3^2 = 3 + \gamma^2\beta^2$,
since $\|e_i\|^2 = \|e'_i\|^2 = -1$ for $i = 1, 2, 3$ and $\gamma^2\beta_i^2 - u_i^2 = \|e'_i\|^2$.

Lemma 3 :

If we consider ${}^t e_1' = (0, 1, 0, 0)$, ${}^t e_1' = (0, 0, 1, 0)$, ${}^t e_1' = (0, 0, 0, 1)$,
$f_1 = M e_1'$, $f_2 = M e_2'$, $f_3 = M e_3'$ *and*
P *the application* $P : {}^t(a, b, c, d) \to {}^t(b, c, d)$
then $\Omega = C^{-1}[P(f_1), P(f_2), P(f_3)]$ *and* $\vec{\beta}$ *is an eigenvector for* $C : C\vec{\beta} = \gamma\vec{\beta}$.

We have $M.\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^t(\gamma\vec{\beta})\Omega \\ C\Omega \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}$ *and* $C\Omega = [P(f_1), P(f_2), P(f_3)]$.

As Maplesoft gives

$$det(C\Omega) = det(C) = \frac{\gamma^2\beta_1^2 + \gamma^2\beta_2^2 + \gamma^2\beta_3^2 + \gamma + 1}{1+\gamma} = 1 + \frac{\gamma^2\vec{\beta}\cdot\vec{\beta}}{1+\gamma} = 1 + \frac{\gamma^2 - 1}{1+\gamma} = \gamma \geq 1.$$

we have $\Omega = C^{-1}[P(f_1), P(f_2), P(f_3)]$

Maplesoft gives

$$C^{-1} = \begin{bmatrix} \frac{\gamma^2\beta_2^2 + \gamma^2\beta_3^2 + \gamma + 1}{\gamma\cdot(\gamma+1)} & -\frac{\gamma^2\beta_1\beta_2}{\gamma\cdot(\gamma+1)} & -\frac{\gamma^2\beta_1\beta_3}{\gamma\cdot(\gamma+1)} \\ -\frac{\gamma^2\beta_1\beta_2}{\gamma\cdot(\gamma+1)} & \frac{\gamma^2\beta_1^2 + \gamma^2\beta_3^2 + \gamma + 1}{\gamma\cdot(\gamma+1)} & -\frac{\gamma^2\beta_2\beta_3}{\gamma\cdot(\gamma+1)} \\ -\frac{\gamma^2\beta_1\beta_3}{\gamma\cdot(\gamma+1)} & -\frac{\gamma^2\beta_2\beta_3}{\gamma\cdot(\gamma+1)} & \frac{\gamma^2\beta_1^2 + \gamma^2\beta_2^2 + \gamma + 1}{\gamma\cdot(\gamma+1)} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\gamma - \gamma\beta_1^2 + 1}{1+\gamma} & -\frac{\gamma\beta_1\beta_2}{1+\gamma} & -\frac{\gamma\beta_1\beta_3}{1+\gamma} \\ -\frac{\gamma\beta_1\beta_2}{1+\gamma} & \frac{\gamma - \gamma\beta_2^2 + 1}{1+\gamma} & -\frac{\gamma\beta_2\beta_3}{1+\gamma} \\ -\frac{\gamma\beta_1\beta_3}{1+\gamma} & -\frac{\gamma\beta_2\beta_3}{1+\gamma} & \frac{\gamma - \gamma\beta_3^2 + 1}{1+\gamma} \end{bmatrix} = Id_{\mathbb{R}^3} - \frac{\gamma\vec{\beta}\,{}^t\vec{\beta}}{(1+\gamma)}$$

because $\gamma^2\beta_2^2 + \gamma^2\beta_3^2 + \gamma + 1 = \gamma^2\beta^2 - \gamma^2\beta_1^2 + \gamma + 1 = \gamma^2 - 1 - \gamma^2\beta_1^2 + \gamma + 1$
$= \gamma\left(1 + \gamma - \gamma\beta_1^2\right)$.

As $M = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta})\Omega \\ \gamma\vec{\beta} & C\Omega \end{bmatrix}$ *is a matrix of Lorentz :*

$$\begin{bmatrix} \gamma & \gamma\overset{t\leftrightarrow}{\beta} \\ \gamma\, {}^t\Omega\cdot\vec{\beta} & {}^t\Omega\cdot C \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 \\ 0 & -1_{\mathbb{R}^3} \end{bmatrix}\cdot\begin{bmatrix} \gamma & \gamma\overset{t\leftrightarrow}{\beta}\,\Omega \\ \gamma\vec{\beta} & C\,\Omega \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & -1_{\mathbb{R}^3} \end{bmatrix}\ \text{then:}$$

$$\begin{bmatrix} -\beta\gamma^2\overset{t\leftrightarrow}{\beta}+\gamma^2 & -C\,\Omega\gamma\overset{t\leftrightarrow}{\beta}+\Omega\gamma^2\overset{t\leftrightarrow}{\beta} \\ -{}^t\Omega C\,\gamma\vec{\beta}+{}^t\Omega\gamma^2\vec{\beta} & {}^t\Omega\gamma^2\vec{\beta}\overset{\leftrightarrow}{\beta}\,\Omega-{}^t\Omega C^2\,\Omega \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & -1_{\mathbb{R}^3} \end{bmatrix}\ \text{and}$$

$$\gamma^2\left(1-\vec{\beta}^2\right)=1\ ;$$
$${}^t\Omega\gamma^2\vec{\beta}\overset{\leftrightarrow}{\beta}\,\Omega-{}^t\Omega C^2\,\Omega=-1_{\mathbb{R}^3}\Rightarrow{}^t\Omega\left(\gamma^2\vec{\beta}\overset{\leftrightarrow}{\beta}{}^t-C^2\right)\Omega=-1_{\mathbb{R}^3}\Rightarrow C^2=1_{\mathbb{R}^3}+\gamma^2\vec{\beta}\overset{\leftrightarrow}{\beta}\ \text{since}\ {}^t\Omega\Omega=1_{\mathbb{R}^3}\ ;$$
$${}^t\Omega C\,\gamma\vec{\beta}={}^t\Omega\gamma^2\vec{\beta}\Rightarrow C\vec{\beta}=\gamma\vec{\beta}.$$

Note :
*For the rest, we recall that the change of basis matrix **M** has its columns equal to the expression of the basis vectors of the new base expressed in the old basis.*

Corollary :

(**0**) *If* **M** *is of Lorentz we have* **M** *symmetric* $\Leftrightarrow \Omega=Id_{\mathbb{R}^3}$.

With the uniqueness of the decomposition we have

$$M=\begin{bmatrix} \gamma & {}^t\left(\gamma\vec{\beta}\right) \\ \gamma\vec{\beta} & C \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}=M.Id_{\mathbb{R}^4}.$$

(**1**) *If* $M=M'.M''$ *is a product of two matrices de Lorentz with*

$$M=\begin{bmatrix} \gamma & {}^t\left(\gamma\vec{\beta}\right)\Omega \\ \gamma\vec{\beta} & C\Omega \end{bmatrix},\ M'=\begin{bmatrix} \gamma' & {}^t\left(\overrightarrow{\gamma'\beta'}\right) \\ \overrightarrow{\gamma'\beta'} & C' \end{bmatrix},\ M''=\begin{bmatrix} \gamma'' & {}^t\left(\overrightarrow{\gamma''\beta''}\right) \\ \overrightarrow{\gamma''\beta''} & C'' \end{bmatrix}$$

$$\text{then}\quad M=\begin{bmatrix} \gamma'\gamma''\left(\overset{t\leftrightarrow}{\beta'}\vec{\beta''}+1\right) & \gamma'\gamma''\overset{t\longleftrightarrow}{\beta''}+\gamma'\overset{t\longleftrightarrow}{\beta'}C'' \\ \gamma'\gamma''\overrightarrow{\beta'}+\gamma''C'\overrightarrow{\beta''} & \gamma'\gamma''\vec{\beta'}\overset{\leftrightarrow}{\beta''}+C'C'' \end{bmatrix}\Rightarrow\gamma=\gamma'\gamma''\left(\overset{\leftrightarrow}{\beta'}\vec{\beta''}+1\right),$$

$$\vec{\beta}=\frac{\overrightarrow{\gamma'\beta'}+C'\overrightarrow{\beta''}}{\gamma'\left(\overset{\leftrightarrow}{\beta'}\vec{\beta''}+1\right)},\ C=Id_{\mathbb{R}^3}+\frac{\gamma^2\vec{\beta}\overset{\leftrightarrow}{\beta}}{(1+\gamma)}\ \text{and}\ \Omega=C^{-1}\left(\gamma'\gamma''\beta'\,{}^t\beta''+C'C''\right).$$

(**2**)
If we have $\Omega=Id_{\mathbb{R}^3}$ *and* $P(f_1)\,//\,P(e_1)$, $P(f_2)\,//\,P(e_2)$, $P(f_3)\,//\,P(e_3)$ *then* $\beta_i\beta_j=0$ *for* $i\neq j$
$\Leftrightarrow\vec{\beta}$ *is parallel to an* e_i $i=1, 2, 3$.

$$\Omega = Id_{\mathbb{R}^3} \Rightarrow C = [P(f_1), P(f_2), P(f_3)] = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$$

since $P(f_1) /\!/ P(e_1)$, $P(f_2) /\!/ P(e_2)$, $P(f_3) /\!/ P(e_3)$,

$\Rightarrow C$ *is diagonal* $\Leftrightarrow \beta_i \beta_j = 0$ *for* $i \neq j \Leftrightarrow \vec{\beta}$ *is parallel to an* e_i $i = 1, 2, 3$.

(3) *If we have* $P(f_1) /\!/ P(e_1), P(f_2) /\!/ P(e_2), P(f_3) /\!/ P(e_3)$ *and*

$\vec{\beta}$ *is parallel to an* e_i $i = 1, 2, 3$ *then* $\Omega = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$:

$\vec{\beta}$ *is parallel to an* e_i $\Rightarrow C$ *is a diagonal matrix* ,

$P(f_1) /\!/ P(e_1), P(f_2) /\!/ P(e_2), P(f_3) /\!/ P(e_3) \Rightarrow C\Omega$ *is also a diagonal matrix*

$\Rightarrow \Omega$ *is diagonal* $\Rightarrow \Omega = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$ *since* Ω *is orthogonal* .

(4) *Let* M *be a matrix of Lorentz such* $\varepsilon = 1$ *and* $P(f_i) = \lambda_i P(e_i)$ *with* $\lambda_i > 0$ *for* $i = 1, 2, 3$, *then* $\Omega = Id_{\mathbb{R}^3}$.

Let us consider $Q = \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix}$ *with* ω *a rotation de* $\mathbb{R}^3$ *such as* $PQ(e_1) = \mu\vec{\beta}$, $\mu > 0$.

Since $P(f_i) = \lambda_i P(e_i)$ *with* $\lambda_i > 0$ *for* $i = 1, 2, 3$ *and* $\vec{\beta}$ *common to both basis* $\mathscr{B}$ *and* $\mathscr{B}'$

$PQ(e'_1) = \mu'\vec{\beta}$, $\mu' > 0$.

We can consider now :

$$\begin{array}{ccc} \mathscr{B} & \xrightarrow{M} & \mathscr{B}' \\ Q\downarrow & & \downarrow Q \\ \mathscr{B}_1 & \xrightarrow{M'} & \mathscr{B}'_1 \end{array}$$ *with* M' *a change of basis matrix of Lorentz.*

If we write $M = \Lambda\tilde{\Omega}$ *with* $\tilde{\Omega} = \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}$ *and* $\Lambda = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta}) \\ \gamma\vec{\beta} & C \end{bmatrix}$:

$$M' = {}^tQMQ = {}^tQ\Lambda\tilde{\Omega}Q = ({}^tQ\Lambda Q)({}^tQ\tilde{\Omega}Q)$$

since M' *and* $({}^tQ\Lambda Q)$ *are symmetric and* ${}^tQ\tilde{\Omega}Q$ *orthogonal then* $\Omega = Id_{\mathbb{R}^3}$ *by uniqueness* .

Corollary :

Let M *be a change of basis matrix of Lorentz such* $\varepsilon = 1$ *associated to* **2** *observers* O *and* O':

$\mathscr{B} = (e_i)_{i=0,3} \xrightarrow{M} \mathscr{B}' = (e'_i)_{i=0,3}$ • *Let's consider another basis* $\mathscr{B}_1 = (f'_i)_{i=0,3}$ *associated to* O':

$\mathscr{B}'$ and $\mathscr{B}_1$ being associated to the same observer, the change of basis matrix is an orthogonal matrix $\boldsymbol{Q}=\begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix}$ with ${}^t\omega\omega = \boldsymbol{Id}_{\mathbb{R}3}$, $\mathscr{B}_1$ chosen as $\boldsymbol{P}(\boldsymbol{f_i}') = \lambda_i \boldsymbol{P}(\boldsymbol{e_i})$ with $\lambda_i > 0$ for $\boldsymbol{i} = 1, 2, 3$ and $\boldsymbol{P}$ the application $\boldsymbol{P} : {}^t(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}) \to {}^t(\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$:

Let's $\boldsymbol{M}'$ the matrix of Lorentz from $\mathscr{B}$ to $\mathscr{B}_1$:

$$\begin{array}{ccc} \mathscr{B} & \xrightarrow{\boldsymbol{M}} & \mathscr{B}' \\ \boldsymbol{M}' \searrow & & \nearrow \boldsymbol{Q} \\ & \mathscr{B}_1 & \end{array}$$

then $\boldsymbol{MX}' = \boldsymbol{M}'\boldsymbol{QX}'$ with $\boldsymbol{M}' = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta}) \\ \gamma\vec{\beta} & \boldsymbol{C} \end{bmatrix}$ and $\boldsymbol{Q} = \begin{bmatrix} 1 & 0 \\ 0 & \omega \end{bmatrix}$ with ${}^t\omega\omega = \boldsymbol{Id}_{\mathbb{R}3}$.

A special case :

Lemma 4 :

Let's assume that $\vec{\beta} = {}^t\left(\beta_1, 0, 0\right)$ *with* $\beta_1 \geq 0$ *• We put* $\delta_{i,j} = 1$ *if* $i = j$ *else* $\delta_{i,j} = 0$.

Let's consider $e_i = \left(\delta_{i,j}\right)$, $j = 0,..., 3$ *and* $M e_i$ *for* $i = 0,..., 3$.

Let's put $\left[{}^t\left(X_0, X_1, X_2, X_3\right)\right] = {}^t\left(X_1, X_2, X_3\right)$.

We assume that $\left[Me_i\right] = \lambda_i\left[e_i\right]$ *with* $\lambda_i > 0$ *for* $i = 1,..., 3$ (1) ,

we have :

$$M = \begin{bmatrix} \gamma & \gamma\beta_1 & 0 & 0 \\ \gamma\beta_1 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Proof :

As $1 + \dfrac{\gamma^2}{(1+\gamma)}\beta^2{}_1 = \dfrac{\left(1+\gamma\right) + \gamma^2\vec{\beta}^2}{(1+\gamma)} = \dfrac{1+\gamma+\gamma^2-1}{(1+\gamma)} = \gamma$,

since $\gamma^2 - 1 = \dfrac{1}{1-\vec{\beta}^2} - 1 = \dfrac{\vec{\beta}^2}{1-\vec{\beta}^2} = \gamma^2\vec{\beta}^2$ *we can write :*

$$M = exp\left(\alpha N\right) \cdot \begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\beta_1 & 0 & 0 \\ \gamma\beta_1 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & \omega_{2,2} & \omega_{2,3} & \omega_{2,4} \\ 0 & \omega_{3,2} & \omega_{3,3} & \omega_{3,4} \\ 0 & \omega_{4,2} & \omega_{4,3} & \omega_{4,4} \end{bmatrix}.$$

$$M e_1 = \begin{bmatrix} \gamma\beta_1\,\omega_{2,2} \\ \gamma\omega_{2,2} \\ \omega_{3,2} \\ \omega_{4,2} \end{bmatrix}, M e_2 = \begin{bmatrix} \gamma\beta_1\,\omega_{2,3} \\ \gamma\omega_{2,3} \\ \omega_{3,3} \\ \omega_{4,3} \end{bmatrix} \text{ and } M e_3 = \begin{bmatrix} \gamma\beta_1\,\omega_{2,4} \\ \gamma\omega_{2,4} \\ \omega_{3,4} \\ \omega_{4,4} \end{bmatrix}$$

then , by (1) : $\begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix} = \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & \omega_{2,2} & 0 & 0 \\ 0 & 0 & \omega_{3,3} & 0 \\ 0 & 0 & 0 & \omega_{4,4} \end{bmatrix}$.

$$As\ {}^t\Omega\Omega = Id_{\mathbb{R}^3},\ M = \begin{bmatrix} \gamma & \gamma\beta_1 & 0 & 0 \\ \gamma\beta_1 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & \varepsilon_1 & 0 & 0 \\ 0 & 0 & \varepsilon_2 & 0 \\ 0 & 0 & 0 & \varepsilon_3 \end{bmatrix} \text{ with } \varepsilon_i = \pm 1.$$

As $\lambda_i > 0$ *and* $\gamma > 0$ *then* $\varepsilon_i = 1$.

We can check ${}^t\boldsymbol{MGM} = \boldsymbol{G}$ *and* $|\boldsymbol{Me}_i| = \lambda_i |\boldsymbol{e}_i|$ ***with*** $\lambda_i > 0$ *for* $\boldsymbol{i = 1,..., 3}$.

In the case where $\boldsymbol{\varepsilon = 1}$ *and* $|\boldsymbol{Me}_i| = \lambda_i |\boldsymbol{e}_i|$ *with* $\lambda_i > 0$ *for* $\boldsymbol{i = 1,..., 3}$,

$\beta_1 \geq 0$, *we have :*

$$M = \begin{bmatrix} \gamma & \gamma\beta_1 & 0 & 0 \\ \gamma\beta_1 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Another case :

<u>Lemma 5</u> *:*

Let $\boldsymbol{M}$ *be a matrix of Lorentz such* $\boldsymbol{\varepsilon = 1}$ *and* $|\boldsymbol{Me}_i| = \lambda_i |\boldsymbol{e}_i|$ *with* $\lambda_i > 0$ *for* $\boldsymbol{i = 1, 2, 3}$,

for any $\vec{\beta}$ *we have* $\boldsymbol{\Omega = Id}_{\mathbb{R}^3}$ *and* $\boldsymbol{M} = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta}) \\ \gamma\vec{\beta} & C \end{bmatrix}$ *with*

$$C = \begin{bmatrix} 1 + \frac{\gamma^2}{(1+\gamma)}\beta^2_1 & \frac{\gamma^2}{(1+\gamma)}\beta_1\beta_2 & \frac{\gamma^2}{(1+\gamma)}\beta_1\beta_3 \\ \frac{\gamma^2}{(1+\gamma)}\beta_2\beta_1 & 1 + \frac{\gamma^2}{(1+\gamma)}\beta^2_2 & \frac{\gamma^2}{(1+\gamma)}\beta_2\beta_3 \\ \frac{\gamma^2}{(1+\gamma)}\beta_3\beta_1 & \frac{\gamma^2}{(1+\gamma)}\beta_3\beta_2 & 1 + \frac{\gamma^2}{(1+\gamma)}\beta^2_3 \end{bmatrix}.$$

Proof :

Let's consider $\boldsymbol{P} = \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix}$ *such* ${}^t\boldsymbol{\Omega\Omega = Id}_{\mathbb{R}^3}$ *and* $\boldsymbol{P}\begin{bmatrix} 0 \\ \vec{\beta} \end{bmatrix} = \lambda \boldsymbol{e}_1$ *with* $\lambda > 0$

and $P\begin{bmatrix} 0 \\ \vec{\beta} \end{bmatrix} = \lambda' e'_1$ since $\vec{\beta}$ is common to both O and O'

and $|Me_1| = \lambda_1 |e_1|$ with $\lambda_1 > 0$.

We can consider now :

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{M} & \mathcal{B}' \\ P\downarrow & & \downarrow P \\ \mathcal{B}_1 & \xrightarrow{M'} & \mathcal{B}'_1 \end{array}$$

With the basis $\mathcal{B}_1 = \left(Pe_0 = e_0, Pe_1, Pe_2, Pe_3\right)$

and $\mathcal{B}'_1 = \left(Pe'_0 = e'_0, Pe'_1, Pe'_2, Pe'_3\right)$ M is represented by $M' = PM\,{}^tP$.

We have $M'Pe_i = PM\,{}^tPPe_i = PM\,e_i = \lambda_i Pe_i$ for $i = 1, 2, 3$.

$${}^tPGP = \begin{bmatrix} 1 & 0 \\ 0 & {}^t\Omega \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -Id_{\mathbb{R}3} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} = G \text{ therefore}$$

M' is of Lorentz.

$$\text{If } W = ct\begin{bmatrix} 1 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}, PW = ct\begin{bmatrix} 1 \\ \Omega\vec{\beta} \end{bmatrix} = ct\begin{bmatrix} 1 \\ \beta'_1 \\ \beta'_2 \\ \beta'_3 \end{bmatrix} = ct\begin{bmatrix} 1 \\ \lambda \\ 0 \\ 0 \end{bmatrix} \text{ and then}$$

$\vec{\beta'} = {}^t\left(\beta'_1, 0, 0\right)$ with $\beta_1 \geq 0$.

$$\text{The lemma } 4 \text{ shows that } M' = \begin{bmatrix} \gamma & \gamma\beta'_1 & 0 & 0 \\ \gamma\beta'_1 & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\text{If } \Lambda(\vec{\beta}) = \begin{bmatrix} \gamma & {}^t(\gamma\vec{\beta}) \\ \gamma\vec{\beta} & C \end{bmatrix} \text{ and } \hat{\Omega} = \begin{bmatrix} 1 & 0 \\ 0 & \Omega \end{bmatrix} \text{ are the factors}$$

of the polar decomposition of M (lemma **3**),

$M' = {}^tPMP = {}^tP\left(\Lambda(\vec{\beta})\,\hat{\Omega}\right)P = \left({}^tP\Lambda(\vec{\beta})P\right)\left({}^tP\hat{\Omega}P\right)$.

M' is symmetric, ${}^tP\Lambda(\vec{\beta})P$ is symmetric and ${}^tP\hat{\Omega}P$ is orthogonal since

${}^t\left({}^tP\hat{\Omega}P\right)\left({}^tP\hat{\Omega}P\right) = Id_{\mathbb{R}4}$ • For short $M' = M' \cdot Id_{\mathbb{R}^4} = \left({}^tP\Lambda(\vec{\beta})P\right)\left({}^tP\hat{\Omega}P\right)$.

As the decomposition is unique ${}^tP\hat{\Omega}P = Id_{\mathbb{R}4}$ $\left(\text{and } M' = \left({}^tP\Lambda(\vec{\beta})P\right)\right)$.

and then $\hat{\Omega} = Id_{\mathbb{R}^4}$.

Eigenvalues and eigenvectors when $\Omega = Id_{\mathbb{R}^3}$.

Lemma 4 :

*With the lemma **1** notations :*

$$N = \begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ x_1 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 \end{bmatrix} \; avec\, \vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \; et\, \vec{X}.\vec{X} = 1 .$$

As the eigenvalues of N are $\{1, -1, 0, 0\}$ and the eigenvectors are :

$$\left\{ \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} -1 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} 0 \\ -x_2 \\ x_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -x_3 \\ 0 \\ x_1 \end{bmatrix} \right\}$$

*If we consider **B** the basis made of thes eigenvectors and the diagonal matrix made of the eigenvalues of N : $N = BDB^{-1}$ •Don't forget that $x_1^2 + x_2^2 + x_3^2 = \vec{X}.\vec{X} = 1$.*

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ x_1 & x_1 & -x_2 & -x_3 \\ x_2 & x_2 & x_1 & 0 \\ x_3 & x_3 & 0 & x_1 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 1 & -1 & 0 & 0 \\ x_1 & x_1 & -x_2 & -x_3 \\ x_2 & x_2 & x_1 & 0 \\ x_3 & x_3 & 0 & x_1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 0 & \dfrac{x_1}{x_1^2 + x_2^2 + x_3^2} & \dfrac{x_2}{x_1^2 + x_2^2 + x_3^2} & \dfrac{x_3}{x_1^2 + x_2^2 + x_3^2} \\ x_1 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 \end{bmatrix}$$

Let's consider now $exp(\alpha N)$, as $exp(\alpha N) = exp(B\alpha DB^{-1}) = Bexp(\alpha D)B^{-1}$ and

thus :

$$exp(\alpha N) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ x_1 & x_1 & -x_2 & -x_3 \\ x_2 & x_2 & x_1 & 0 \\ x_3 & x_3 & 0 & x_1 \end{bmatrix} . \begin{bmatrix} e^{\alpha} & 0 & 0 & 0 \\ 0 & e^{-\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} 1 & -1 & 0 & 0 \\ x_1 & x_1 & -x_2 & -x_3 \\ x_2 & x_2 & x_1 & 0 \\ x_3 & x_3 & 0 & x_1 \end{bmatrix}^{-1}$$

However $\gamma = ch(\alpha) \Leftrightarrow \alpha = ln\left(\gamma + \sqrt{\gamma^2 - 1}\right) \Rightarrow e^{\alpha} = \gamma + \sqrt{\gamma^2 - 1,}$

as $\gamma^2 - 1 = \beta^2\gamma^2 \Rightarrow e^{\alpha} = \gamma(1+\beta) = \dfrac{(1+\beta)}{\sqrt{(1+\beta)(1-\beta)}} = \sqrt{\dfrac{1+\beta}{1-\beta}}$ *and*

$e^{-\alpha} = \gamma(1-\beta) = \sqrt{\dfrac{1-\beta}{1+\beta}}$ *with* $\beta = \sqrt{\vec{\beta}\cdot\vec{\beta}}$.

We assume that $\beta \neq 0$ *if not* $exp(\alpha N) = Id$.

As $coth(\alpha) = \beta^{-1}$ *and* $\vec{X} = coth(\alpha)\vec{\beta}$ **:**

We assume that $\beta \neq 0$

$$exp(\alpha N) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ \beta^{-1}\cdot\beta_1 & \beta^{-1}\cdot\beta_1 & -\beta^{-1}\cdot\beta_2 & -\beta^{-1}\cdot\beta_3 \\ \beta^{-1}\cdot\beta_2 & \beta^{-1}\cdot\beta_2 & \beta^{-1}\cdot\beta_1 & 0 \\ \beta^{-1}\cdot\beta_3 & \beta^{-1}\cdot\beta_3 & 0 & \beta^{-1}\cdot\beta_1 \end{bmatrix} \cdot \begin{bmatrix} \gamma\cdot(1+\beta) & 0 & 0 & 0 \\ 0 & \gamma\cdot(1-\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 & -1 & 0 & 0 \\ \beta^{-1}\cdot\beta_1 & \beta^{-1}\cdot\beta_1 & -\beta^{-1}\cdot\beta_2 & -\beta^{-1}\cdot\beta_3 \\ \beta^{-1}\cdot\beta_2 & \beta^{-1}\cdot\beta_2 & \beta^{-1}\cdot\beta_1 & 0 \\ \beta^{-1}\cdot\beta_3 & \beta^{-1}\cdot\beta_3 & 0 & \beta^{-1}\cdot\beta_1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \gamma & \dfrac{\gamma\beta_1\beta^2}{\beta_1^2+\beta_2^2+\beta_3^2} & \dfrac{\gamma\beta^2\beta_2}{\beta_1^2+\beta_2^2+\beta_3^2} & \dfrac{\gamma\beta_3\beta^2}{\beta_1^2+\beta_2^2+\beta_3^2} \\ \beta_1\gamma & \dfrac{\gamma\beta_1^2+\beta_2^2+\beta_3^2}{\beta_1^2+\beta_2^2+\beta_3^2} & \dfrac{\beta_2\beta_1(\gamma-1)}{\beta_1^2+\beta_2^2+\beta_3^2} & \dfrac{\beta_3\beta_1(\gamma-1)}{\beta_1^2+\beta_2^2+\beta_3^2} \\ \beta_2\gamma & \dfrac{\beta_2\beta_1(\gamma-1)}{\beta_1^2+\beta_2^2+\beta_3^2} & \dfrac{\gamma\beta_2^2+\beta_1^2+\beta_3^2}{\beta_1^2+\beta_2^2+\beta_3^2} & \dfrac{\beta_2\beta_3(\gamma-1)}{\beta_1^2+\beta_2^2+\beta_3^2} \\ \beta_3\gamma & \dfrac{\beta_3\beta_1(\gamma-1)}{\beta_1^2+\beta_2^2+\beta_3^2} & \dfrac{\beta_2\beta_3(\gamma-1)}{\beta_1^2+\beta_2^2+\beta_3^2} & \dfrac{\gamma\beta_3^2+\beta_1^2+\beta_2^2}{\beta_1^2+\beta_2^2+\beta_3^2} \end{bmatrix}$$

$$
= \begin{bmatrix}
\gamma & \gamma\cdot\beta_1 & \gamma\cdot\beta_2 & \gamma\cdot\beta_1 \\
\gamma\cdot\beta_1 & 1+\frac{\beta_1^2(\gamma-1)}{\beta^2} & \frac{\beta_2\beta_1(\gamma-1)}{\beta^2} & \frac{\beta_3\beta_1(\gamma-1)}{\beta^2} \\
\gamma\cdot\beta_2 & \frac{\beta_2\beta_1(\gamma-1)}{\beta^2} & 1+\frac{\beta_2^2(\gamma-1)}{\beta^2} & \frac{\beta_2\beta_3(\gamma-1)}{\beta^2} \\
\gamma\cdot\beta_3 & \frac{\beta_3\beta_1(\gamma-1)}{\beta^2} & \frac{\beta_2\beta_3(\gamma-1)}{\beta^2} & 1+\frac{\beta_3^2(\gamma-1)}{\beta^2}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\gamma & \gamma\cdot\beta_1 & \gamma\cdot\beta_2 & \gamma\cdot\beta_3 \\
\gamma\cdot\beta_1 & 1+\frac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_1 & \frac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_2 & \frac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_3 \\
\gamma\cdot\beta_2 & \frac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_1 & 1+\frac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_2 & \frac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_3 \\
\gamma\cdot\beta_3 & \frac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_1 & \frac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_2 & 1+\frac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_3
\end{bmatrix}
$$

because $1 \Big/ \left((1+\gamma)\gamma^{2}\right) = \frac{(\gamma-1)}{\beta^2}$.

We can verify :

$$
\begin{bmatrix}
\gamma & \gamma\cdot\beta_1 & \gamma\cdot\beta_2 & \gamma\cdot\beta_3 \\
\gamma\cdot\beta_1 & 1+\frac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_1 & \frac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_2 & \frac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_3 \\
\gamma\cdot\beta_2 & \frac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_1 & 1+\frac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_2 & \frac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_3 \\
\gamma\cdot\beta_3 & \frac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_1 & \frac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_2 & 1+\frac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_3
\end{bmatrix}
\cdot
\begin{bmatrix}
1 \\
\beta^{-1}\cdot\beta_1 \\
\beta^{-1}\cdot\beta_2 \\
\beta^{-1}\cdot\beta_3
\end{bmatrix}
=
$$

$$\begin{bmatrix} \dfrac{\gamma\left(\beta_1^2+\beta_2^2+\beta_3^2+\beta\right)}{\beta} \\ \dfrac{\beta_1\left(1+\left(\beta_1^2+\beta_2^2+\beta_3^2+\beta\right)\gamma^2+(\beta+1)\gamma\right)}{(1+\gamma)\beta} \\ \dfrac{\beta_2\left(1+\left(\beta_1^2+\beta_2^2+\beta_3^2+\beta\right)\gamma^2+(\beta+1)\gamma\right)}{(1+\gamma)\beta} \\ \dfrac{\beta_3\left(1+\left(\beta_1^2+\beta_2^2+\beta_3^2+\beta\right)\gamma^2+(\beta+1)\gamma\right)}{(1+\gamma)\beta} \end{bmatrix} = \gamma(1+\beta)\begin{bmatrix} 1 \\ \beta^{-1}\cdot\beta_1 \\ \beta^{-1}\cdot\beta_2 \\ \beta^{-1}\cdot\beta_3 \end{bmatrix} \text{ because}$$

$$1+\left(\beta_1^2+\beta_2^2+\beta_3^2+\beta\right)\gamma^2+(\beta+1)\gamma = 1+\beta\cdot(1+\beta)\gamma^2+(\beta+1)\gamma$$
$$=1+(\beta+1)\gamma(1+\gamma).$$

$$\text{thus } \frac{\beta_1\left(1+\left(\beta_1^2+\beta_2^2+\beta_3^2+\beta\right)\gamma^2+(\beta+1)\gamma\right)}{(1+\gamma)\beta} = \frac{\beta_1(1+(\beta+1)\gamma(1+\gamma))}{(1+\gamma)\beta} \text{ however}$$

$$\gamma(1+\beta)\beta^{-1}\cdot\beta_1 = \frac{\beta_1\left(1+\left(\beta^2+\beta\right)\gamma^2+(\beta+1)\gamma\right)}{(1+\gamma)\beta} \Leftrightarrow \gamma(1+\beta)\cdot\beta_1(1+\gamma) = \beta_1\left(1+\beta\left(\gamma+\gamma^2\right)+\gamma^2-1+\gamma\right)$$

because $\gamma^2\beta^2=\gamma^2-1$ et $\beta_1\left(\beta\left(\gamma+\gamma^2\right)+\gamma^2+\gamma\right)=\beta_1\left(\gamma+\gamma^2\right)(1+\beta)=\beta_1(1+\gamma)\gamma(1+\beta)$.

In the same way :

$$\begin{bmatrix} \gamma & \gamma\cdot\beta_1 & \gamma\cdot\beta_2 & \gamma\cdot\beta_3 \\ \gamma\cdot\beta_1 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_1 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_3 \\ \gamma\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_1 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_3 \\ \gamma\cdot\beta_3 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_1 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_2 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ \beta^{-1}\cdot\beta_1 \\ \beta^{-1}\cdot\beta_2 \\ \beta^{-1}\cdot\beta_3 \end{bmatrix} =$$

$$\begin{bmatrix} -\dfrac{\gamma\left(-\beta_1^2-\beta_2^2-\beta_3^2+\beta\right)}{\beta} \\ -\dfrac{\left(-1+\left(-\beta_1^2-\beta_2^2-\beta_3^2+\beta\right)\gamma^2+(\beta-1)\gamma\right)\beta_1}{(1+\gamma)\beta} \\ -\dfrac{\left(-1+\left(-\beta_1^2-\beta_2^2-\beta_3^2+\beta\right)\gamma^2+(\beta-1)\gamma\right)\beta_2}{(1+\gamma)\beta} \\ -\dfrac{\left(-1+\left(-\beta_1^2-\beta_2^2-\beta_3^2+\beta\right)\gamma^2+(\beta-1)\gamma\right)\beta_3}{(1+\gamma)\beta} \end{bmatrix} = \gamma(1-\beta)\begin{bmatrix} -1 \\ \beta^{-1}\cdot\beta_1 \\ \beta^{-1}\cdot\beta_2 \\ \beta^{-1}\cdot\beta_3 \end{bmatrix}.$$

We can point out that the 2 first eigenvectors are light vectors .

$$\begin{bmatrix} \gamma & \gamma\cdot\beta_1 & \gamma\cdot\beta_2 & \gamma\cdot\beta_3 \\ \gamma\cdot\beta_1 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_1 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_3 \\ \gamma\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_1 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_3 \\ \gamma\cdot\beta_3 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_1 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_2 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_3 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ -\beta^{-1}\cdot\beta_2 \\ \beta^{-1}\cdot\beta_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -\beta^{-1}\cdot\beta_2 \\ \beta^{-1}\cdot\beta_1 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} \gamma & \gamma\cdot\beta_1 & \gamma\cdot\beta_2 & \gamma\cdot\beta_3 \\ \gamma\cdot\beta_1 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_1 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_1\cdot\beta_3 \\ \gamma\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_1 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_2 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_2\cdot\beta_3 \\ \gamma\cdot\beta_3 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_1 & \dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_2 & 1+\dfrac{\gamma^2}{(1+\gamma)}\cdot\beta_3\cdot\beta_3 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ -\beta^{-1}\cdot\beta_3 \\ 0 \\ \beta^{-1}\cdot\beta_1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -\beta^{-1}\cdot\beta_3 \\ 0 \\ \beta^{-1}\cdot\beta_1 \end{bmatrix}.$$

For short the eigenvalues are $\{\gamma(1+\beta), \gamma(1-\beta), 1, 1\}$ *and*

the eingeinvectors $\left\{ \begin{bmatrix} 1 \\ \beta^{-1}\cdot\beta_1 \\ \beta^{-1}\cdot\beta_2 \\ \beta^{-1}\cdot\beta_3 \end{bmatrix}, \begin{bmatrix} -1 \\ \beta^{-1}\cdot\beta_1 \\ \beta^{-1}\cdot\beta_2 \\ \beta^{-1}\cdot\beta_3 \end{bmatrix}, \begin{bmatrix} 0 \\ -\beta^{-1}\cdot\beta_2 \\ \beta^{-1}\cdot\beta_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\beta^{-1}\cdot\beta_3 \\ 0 \\ \beta^{-1}\cdot\beta_1 \end{bmatrix} \right\}$

Eigenvalues of a product of 2 symmetric matrices :

If A *and* B *are real symmetric matrices such that the eigenvalues of* A *are strictly positive and those of* B *positive then the product* AB *is also* $\mathbb{R}-$*diagonalizable .*

(*We remind that if* A *is symmetric with positive eigenvalue then there exists a single matrice symmetric with eigenvalues positive* $A^{\frac{1}{2}}$ *such as* $\left(A^{\frac{1}{2}} \bullet A^{\frac{1}{2}}\right) = A$).

Proof :

We can write $AB = A^{\frac{1}{2}}\left(A^{\frac{1}{2}} B A^{\frac{1}{2}}\right)A^{-\frac{1}{2}}$ *, thus* AB ***and*** $A^{\frac{1}{2}} B A^{\frac{1}{2}}$ *have the same eigenvalues .*

However ${}^{t}\left(A^{\frac{1}{2}} B A^{\frac{1}{2}}\right) = \left(A^{\frac{1}{2}} B A^{\frac{1}{2}}\right)$ *thus* $\left(A^{\frac{1}{2}} B A^{\frac{1}{2}}\right)$ *is symmetric therefore* $\mathbb{R}$*-diagonalizable .*

Note : If $\Omega \neq Id_{\mathbb{R}^3}$ M *may have no real eigenvalues for example* $\vec{\beta} = \vec{0}$ *and* Ω *being a rotation .*

(5) **Space − time vectors properties :**

*In a Minkowski space, the quadratic form associated with this space makes it possible to classify the vectors into **3** categories that we are going to study.*

Vectors in a Minkowski − space

(*J-M. Souriau.*"Calcul Linéaire " •*PUF 1964* .)

We consider the vector space $\mathbb{R}^4$ *of* **4** *dimensions with the quadratic form of Lorentz :*

$$\Phi(X) = x_1^2 - \sum_{i=2}^{4} x_i^2 \text{ where } {}^bX = (x_1,..., x_4) \text{ , whose matrix is :}$$

$$G = \begin{bmatrix} 1 & {}^t0 \\ 0 & -Id_{\mathbb{R}^3} \end{bmatrix} \text{ where } \mathbf{0} \text{ is the null column of } \mathbb{R}^3 .$$

We give $\mathbb{R}^4$ *a* Φ*-orthonormal basis .*

We want to describe the sets of vectors $\mathbb{R}^4$ *defined according **to** the sign of* $\Phi(X)$.

We distinguish 3 subsets :

$E_+ = \{X \in \mathbb{R}^4 / \Phi(X) > 0\} \cup \{0\}$ *: time vectors,*

$E_- = \{X \in \mathbb{R}^4 / \Phi(X) < 0\} \cup \{0\}$ *: space vectors,*

$E_0 = \{X \in \mathbb{R}^4 / \Phi(X) = 0\}$ *: isotropic vectors .*

Whether or not to add $\{0\}$ *to* E_+ *and to* E_- *varies according to different authors.*

We notice that the nature of vectors is independent of the chosen basis of $\mathbb{R}^4$ Φ *- orthonormed .*
If we note P *the matrix of passage from the basis* $\mathscr{B}$ *to the basis* $\mathscr{B}'$ *,*
if P *is a matrix of Lorentz;* ${}^tPGP = G$ *,* $X = PX'$*,* $Y = PY'$ *then*

$${}^tX\,GX = ({}^tX'\,{}^tP)\,G(PX') = {}^tX'({}^tPGP)X' = {}^tX'GX' .$$

Lemma 1 :

We consider the vector space $\mathbb{R}^4$ *of* **4** *dimensions with the quadratic form of Lorentz.*
In $\mathbb{R}^4$ *there is a* **3** *− dimensional subspace of space vectors .*

Proof :

Consider for example : $\{X \in \mathbb{R}^4 / x_1 = 0\}$.

Lemma 2 :

We consider the vector space $\mathbb{R}^4$ *with the quadratic form of Lorentz.*
There are no time vectors subspace of dimension $>= 2$.

Proof :

Because otherwise there exists a subspace F *of dimension at least equal to* **2** *of time vectors .*
As there exists a subspace G of space vectors of dimension **3** *and as* $F \cap G = \{0\}$ *and therefore* $\dim(\mathbb{R}^4) \geq 5$.

Lemma 3 :

We consider the vector space with the quadratic form of Lorentz.
Let **2** *vectors of* $\mathbb{R}^4$ X *and* $Y \neq \{0\}$ *such that* ${}^tXGX \geq 0$ *,* ${}^tYGY \geq 0$ *and* ${}^tXGY = 0$
then X *and* Y *are collinear and isotropic.*

Proof

We consider X *and* Y **2** *non-zero time vectors.*
If they are independent they generate a subspace of dimension **2** *of positive vectors :*

If $Z = \lambda X + \mu Y$ *then* ${}^tZGZ = {}^t(\lambda X + \mu Y)\,G(\lambda X + \mu Y) = \lambda^2\,{}^tXGX + \mu^2\,{}^tYGY \geq 0$.

It is impossible therefore : $\exists\, \lambda \neq 0 \in \mathbb{R}$ *such that*

$X=\lambda Y$ *and then* $: 0={}^tXGY=\lambda {}^tY\,GY=\frac{1}{\lambda}{}^tX\,GX$.

Lemma 4 :

(**1**) *Any non* $-$ *zero vector* **X** *orthogonal to a non - zero time vector* **Y** $({}^tXGY=0)$ *is a space vector.*
(**2**) **2** *non* $-$ *zero isotropic independent vectors* **X** *and* **Y** *are never orthogonal.*

Proof :

(**1**) *If* **X** *was a non* $-$ *zero time vector :* ${}^tXGX>0$ *and* **Y** *such that* ${}^tXGX>0$ *then* ${}^tYGY<0$ *because otherwise* ${}^tYGY\geq 0$ ***and*** *as* ${}^tXGX>0$ *and* ${}^tXGY=0$ *, according to* ***lemma 3 X*** *would be isotropic, which is contradictory.*

(**2**) *Because otherwise* $0={}^tXGY={}^tXGX={}^tYGY$ *according to* ***lemma 5 X*** *and* **Y** *are linearly dependent.*

Definition : *For any time vector* $({}^tXGX\geq 0)$ *we set* $\|X\|_G=\sqrt{{}^tXGX}$.

Lemma 5 :

Let φ *be a bilinear symmetrical nondegenerate form on a vector space* **E** *of dimension* **n**, *then for any base* $\mathcal{B}(e_1, ..., e_n)$*, if we consider the matrix representing* φ *in* $\mathcal{B}$

$Q=(\varphi(e_i, e_j))$*, then the determinant of* **Q** *is of the sign of* $(-1)^{n-p}$
where **p** *is the positive index of inertia of* φ.

Proof :

There exists a base $\mathcal{B}'(e'_1, ..., e'_n)$ *where the matrix representing* φ *is the form :*

$$Q'=\begin{bmatrix} Id_{\mathbb{R}^p} & 0 \\ 0 & -Id_{\mathbb{R}^q}\end{bmatrix}, \text{then } \det(Q')=(-1)^{n-p}.$$

Let **S** *be the matrix of passage from the basis* $\mathcal{B}'$ *to the basis* $\mathcal{B}$ *we have*
$$\det(Q)=\det({}^tSQ'S)=\det(Q')(\det(S))^2 \Rightarrow \operatorname{sign}(\det(Q))=(-1)^{n-p}.$$

Lemma 6 : *Cauchy* $-$ *Schwartz's counter inequality .*
We have $|{}^tXGY|\geq \|X\|_G\|Y\|_G$ *for any time or isotropic vectors.*

Proof :

Let us consider the matrix **S** *made up of the* **2** *columns* $X\in\mathbb{R}^4$ *and* $Y\in\mathbb{R}^4$ $: S=[X, Y]$.
We assume that **X** *and* **Y** *are non* $-$ *zero time vectors because if one of them is isotropic or zero the `inequality is obvious.*

$$\text{We have } {}^tSGS=\begin{bmatrix} {}^tX \\ {}^tY\end{bmatrix}G[X, Y]=\begin{bmatrix} {}^tXGX & {}^tXGY \\ {}^tYGX & {}^tYGY\end{bmatrix},$$

and $\det({}^tSGS)=\|X\|_G^2\|Y\|_G^2-({}^tXGY)^2$.

If $\det({}^tSGS)=0$ *the lemma is proved .*
If $\det({}^tSGS)\neq 0$ *and if* $X=\lambda Y$, $\lambda\in\mathbb{R}^*$,

$$\text{then } \det({}^tSGS)=\begin{vmatrix} \lambda^2\,{}^tYGY & \lambda\,{}^tYGY \\ \lambda\,{}^tYGY & {}^tYGY\end{vmatrix}=0$$

It's impossible therefore **X** *and* **Y** *are non coplanar :* **X** *and* **Y** *form a base* $S=[X, Y]$ *of a vector subspace* **F** *of* $\mathbb{R}^4$ *of dimension* **2**.

As $^tSGS = \begin{bmatrix} ^tXGX & ^tXGY \\ ^tYGX & ^tYGY \end{bmatrix}$, ^tSGS *defines a form, bilinear and symmetrical* Ψ

on F *by* $\Psi(u, v) = {}^tU\,{}^tSGSV$ *with* $u = SU$ *and* $v = SV$.

Let φ *be thebilinear form defined` on* $\mathbb{R}^4$ *whose matrix is* G.

We immediately check that $\varphi_{/F}$ *is also a bilinear symmetrical form on* F.

Let us show that $\varphi_{/F}$ *is regular, that is :*

$F^{\perp} = \{u \in F / \varphi_{/F}(u, v) = 0\ ,\ \forall v \in F\} = \{0\}$.

S*, being a basis of* F*, is a bijective application of* $\mathbb{R}^2$ *on* F.

If G *is the representation of* φ $\mathbb{R}^4$, ^tSGS *is the representation of* $\varphi_{/F}$ *in* F

provided with the base S *: let* u *and* v *2 vectors of* F*;*

we have if $\tilde{U}$ *and* $\tilde{V}$ *the representations of* u *and* v *in* $\mathbb{R}^4$*,*

U *and* V *in F with the base* S *and* $\tilde{U} = SU$, $\tilde{V} = SV$ *:*

$\varphi(u, v) = \varphi_{/F}(u, v) = {}^t\tilde{U}\,G\,\tilde{V} = {}^tU\,{}^tSGSV.$

So let u $u \in F^{\perp}$ *and* $v \in F$, $u = SU$ *and* $v = SV$ *with* U *and* V *element of* $\mathbb{R}^2$.

We have $\varphi_{/F}(u, v) = {}^tV\,{}^tSGSU = 0 \quad \forall V \in \mathbb{R}^2$ *and therefore* $^tSGSU = 0$

as $\det(SGS) \neq 0$ *we have* $U = 0$.

$\varphi_{/F}$ *is indeed a bilinear symmetric regular form on* F.

The **lemma 5** *applies :* $\mathrm{sign}(\det(^tSGS)) = (-1)^{2-p}$, p *the inertia index of* $\varphi_{/F}$

but F *being of dimension* 2*, according to the* **lemma 2** *there is no subspace of dimension* $>$
$= 2$ *of time vectors*

and F *contains* X*, time vector.*

$\varphi_{/F}$ *being a bilinear symmetrical regular form on F,* $\varphi_{/F}$ *can be represented*

in a $\varphi_{/F}$ *– orthogonal basis by a diagonal matrix composed` of* 1, -1 *and* 0.

The nature of the vectors remaining unchanged, the only possibility is therefore 1 *and* -1.

So the only possibility for p *is* $p = 1$.

Therefore $\mathrm{sign}(\det(^tSGS)) = -1$ **et** *donc* $\|X\|_G^2\|Y\|_G^2 < (^tXGY)^2$.

Note;

We recall that the set of time vectors do not form a vector subspace

take for example; $X = {}^t(4, 1, 1, 1)$ **et** $Y = {}^t(-4, 1, 1, 1)$ **et** $X + Y = {}^t(0, 2, 2, 2)$.

Lemme *7 :*

Let X, Y *and* Z *3 time or isotrope - vectors we then have :*

$$(^tXGY)(^tYGZ)(^tZGX) \geq (^tXGX)(^tYGY)(^tZGZ)$$

Proof :

Let's consider $S = [X, Y, Z]$*, we have :*

$$ {}^{t}SGS = \begin{bmatrix} {}^{t}X \\ {}^{t}Y \\ {}^{t}Z \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -Id_{\mathbb{R}^3} \end{bmatrix} [X, Y, Z] = \begin{bmatrix} X_1 & -X_2 & -X_3 & -X_4 \\ Y_1 & -Y_2 & -Y_3 & -Y_3 \\ Z_1 & -Z_2 & -Z_3 & -Z_4 \end{bmatrix} \begin{bmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \\ X_4 & Y_3 & Z_4 \end{bmatrix} $$

$$ = \begin{bmatrix} {}^{t}XGX & {}^{t}XGY & {}^{t}XGZ \\ {}^{t}YGX & {}^{t}YGY & {}^{t}YGZ \\ {}^{t}ZGX & {}^{t}ZGY & {}^{t}ZGZ \end{bmatrix} $$

and $\det({}^{t}SGS) = {}^{t}XGX[{}^{t}YGY \cdot {}^{t}ZGZ - {}^{t}YGZ \cdot {}^{t}ZGY]$

$$ -{}^{t}XGY[{}^{t}YGX \cdot {}^{t}ZGZ - {}^{t}YGZ \cdot {}^{t}ZGX] $$
$$ + {}^{t}XGZ[{}^{t}YGX \cdot {}^{t}ZGY - {}^{t}YGY \cdot {}^{t}ZGX] $$
$$ = {}^{t}XGX \cdot {}^{t}YGY \cdot {}^{t}ZGZ - {}^{t}XGX \cdot {}^{t}YGZ \cdot {}^{t}ZGY $$
$$ -{}^{t}XGY \cdot {}^{t}YGX \cdot {}^{t}ZGZ + {}^{t}XGY \cdot {}^{t}YGZ \cdot {}^{t}ZGX $$
$$ + {}^{t}XGZ \cdot {}^{t}YGX \cdot {}^{t}ZGY - {}^{t}XGZ \cdot {}^{t}YGY \cdot {}^{t}ZGX $$
$$ = {}^{t}XGX \cdot {}^{t}YGY \cdot {}^{t}ZGZ - {}^{t}XGX({}^{t}YGZ)^2 $$
$$ -{}^{t}YGY \cdot ({}^{t}XGZ)^2 - {}^{t}ZGZ\,({}^{t}XGY)^2 + 2 \cdot {}^{t}XGY \cdot {}^{t}YGZ \cdot {}^{t}ZGX ; $$

But $|{}^{t}XGY| \geq \sqrt{{}^{t}XGX}\sqrt{{}^{t}YGY}$, $|{}^{t}YGZ| \geq \sqrt{{}^{t}YGY}\sqrt{{}^{t}ZGZ}$, $|{}^{t}ZGX| \geq \sqrt{{}^{t}ZGZ}\sqrt{{}^{t}XGX}$,
then :

$$ -{}^{t}XGX({}^{t}YGZ)^2 \; -{}^{t}YGY \cdot ({}^{t}XGZ)^2 - {}^{t}ZGZ\,({}^{t}XGY)^2 $$
$$ \leq -{}^{t}XGX \cdot {}^{t}YGY \cdot {}^{t}ZGZ - {}^{t}YGY \cdot {}^{t}ZGZ \cdot {}^{t}XGX - {}^{t}ZGZ \cdot {}^{t}XGX \cdot {}^{t}YGY $$
$$ = -3 \cdot {}^{t}XGX \cdot {}^{t}YGY \cdot {}^{t}ZGZ , $$

and then $\det({}^{t}SGS) \leq -2 \cdot {}^{t}XGX \cdot {}^{t}YGY \cdot {}^{t}ZGZ + 2\,{}^{t}XGY \cdot {}^{t}YGZ \cdot {}^{t}ZGX.$

Considering the sign of $\det({}^{t}SGS)$ *:*

If $\det({}^{t}SGS) = 0$ *the lemma is proved .*

If $\det({}^{t}SGS) \neq 0$ *and if* $X = \lambda Y + \mu Z$, $\lambda \in \mathbb{R}^*$, $\mu \in \mathbb{R}^*$,

we have $\det({}^{t}SGS) = \det \begin{bmatrix} {}^{t}XGX & {}^{t}XGY & {}^{t}XGZ \\ {}^{t}YGX & {}^{t}YGY & {}^{t}YGZ \\ {}^{t}ZGX & {}^{t}ZGY & {}^{t}ZGZ \end{bmatrix} = 0$ *because* $(\lambda^{t}Y + \mu^{t}Z)GX = \lambda^{t}YGX + \mu^{t}ZGX$

it's impossible therefore X , Y *et* Z *are non − collinear and form the basis of a vector subspace F of* $\mathbb{R}^4$ *of dimension* 3.

As ${}^{t}SGS = \begin{bmatrix} {}^{t}XGX & {}^{t}XGY & {}^{t}XGZ \\ {}^{t}YGX & {}^{t}YGY & {}^{t}YGZ \\ {}^{t}ZGX & {}^{t}ZGY & {}^{t}ZGZ \end{bmatrix}$, ${}^{t}SGS$ *is a bilinear symetrical form on* ***F.***

By making a similar reasoning ***to*** *that of the previous lemma we find that*
$sign(\det({}^{t}SGS)) = (-1)^{3-1} = 1$.

Therefore $\det({}^{t}SGS) > 0$ *and the lemma is proved.*

Lemma 8 :

We consider $\mathbb{R}^4$ provided with the quadratic form defined by $G = \begin{bmatrix} 1 & 0 \\ 0 & -Id_{\mathbb{R}^3} \end{bmatrix}$,

then the union of non $-$ zero time vectors and the isotropic vectors are divided into **2** opposite classes $\mathcal{C}_1$ and $\mathcal{C}_2$ and if **X** and **Y** are non - zero time vectors we have :

$^tXGY \geq 0 \Leftrightarrow$ **X** and **Y** belong to the same class ,

$^tXGY \leq 0 \Leftrightarrow$ **X** and **Y** belong to opposite classes .

Two vectors **X** and **Y** belonging to a same class check $^tXGY > 0$ unless they are isotropic and parallel. In this case they belong to the same class if their ratio is a strictly positive number.

Proof:

Let X_0 be an arbitrary non $-$ zero time vector. A non $-$ zero time vector **Y** is never orthogonal to X_0 because otherwise by **lemma 4, Y** would be a space vector so $^tYGX_0 > 0$ or $^tYGX_0 < 0$.

We say that $Y \in \mathcal{C}_1$ in the first case otherwise $Y \in \mathcal{C}_2$. The classes are opposite : if $Y \in \mathcal{C}_1$ then $-Y \in \mathcal{C}_2$ • The **lemm**a 7 shows that :

$$\left({}^tX_0GY\right)\left({}^tYGZ\right)\left({}^tZGX_0\right) \geq \left({}^tX_0GX_0\right)\left({}^tYGY\right)\left({}^tZGZ\right) \geq 0,\ \forall Y,\ \forall Z \text{ time vectors}.$$

If **Y** and **Z** belong

$-$ to the same class $\mathcal{C}_1$: $^tX_0GY \geq 0$ et $^tZGX_0 \geq 0 \Rightarrow {}^tYGZ \geq 0$,

$-$ to the same class $\mathcal{C}_2$: $^tX_0GY \leq 0$ et $^tZGX_0 \leq 0 \Rightarrow {}^tYGZ \geq 0$

$-$ to different classes : $^tX_0GY \leq 0$ et $^tZGX_0 \geq 0 \Rightarrow {}^tYGZ \leq 0$ or

$^tX_0GY \geq 0$ et $^tZGX_0 \leq 0 \Rightarrow {}^tYGZ \leq 0$.

Conversely : If $^tYGZ \geq 0$

either $^tX_0GY \geq 0$ and $^tZGX_0 \geq 0 \Rightarrow$ **Y** and **Z** belong to the same class $\mathcal{C}_1$,

or $^tX_0GY \leq 0$ and $^tZGX_0 \leq 0 \Rightarrow$ **Y** and **Z** belong to the same class $\mathcal{C}_2$,

in the same way if $^tYGZ \leq 0 \Rightarrow$ **Y** et **Z** belong to different classes .

The last part of the lemma is a direct consequence of **lemma 3**.

Note : In an arbitrary way the elements of one of the **2** classes are called vectors of future, the elements of the other vectors of past.

Lemme 9 :

Soient **X** et **Y** **2** vecteurs de temps ou isotropes .

Si **X** et **Y** appartiennent à la même classe , leur somme est encore un vecteur de la même classe , et ils vérifient la contre $-$ inégalité triangulaire si on note $\|X\|_G = \sqrt{{}^tXX}$ si $^tXX \geq 0$:

$\|X+Y\|_G \geq \|X\|_G + \|Y\|_G$.

Proof:

From the **lemma 8** $^tXGY \geq 0$ and from the **lemm**a **6** $\left|{}^tXGY\right| \geq \|X\|_G\|Y\|_G$ and then

$XGY \geq \|X\|_G\|Y\|_G$, therefore :

$\|X+Y\|_G^2 = {}^t\left(X+Y\right)G\left(X+Y\right) = \|X\|_G^2 + \|Y\|_G^2 + 2{}^tXGY$

$\geq \|X\|_G^2 + \|Y\|_G^2 + + 2\ \|Y\|_G\|X\|_G = \left(\|X\|_G + \|X\|_G\right)^2$.

If **V** is a vector taken in the same class of **X** et **Y** , we have :

${}^{t}XGV \geq 0$ *and* ${}^{t}YGV \geq 0 \Rightarrow {}^{t}(X+Y)GV \geq 0$ *and* $X+Y$ *belons to the class of* V *therefore* **to** *the class of* **X and Y**.

Note :

(1) *If* **X** *and* **Y** *are* **2** *vectors of class different the sum can be of any kind :*

If $X = {}^{t}(2, 1, 1, 1)$, ${}^{t}XGX = 1$ *and* $Y = {}^{t}(-3, 1, 1, 1)$, ${}^{t}YGY = 6$, *then* ${}^{t}XGY = -3$,

$X+Y = {}^{t}(-1, 2, 2, 2)$, ${}^{t}(X+Y)G(X+Y) = -11$ (*space vector*) **.**

If $X = {}^{t}(4, 1, 1, 1)$, ${}^{t}XGX = 13$ *and* $Y = {}^{t}(-1, 0, 0, 0)$, ${}^{t}YGY = 1$, *then* ${}^{t}XGY = -4$,

$X+Y = {}^{t}(3, 1, 1, 1)$, ${}^{t}(X+Y)G(X+Y) = 6$ (*time vector*).

If $X = {}^{t}(3, 1, 1, 1)$, ${}^{t}XGX = 6$ *and* $Y = {}^{t}\left(-2, \frac{\sqrt{3}}{3} - 1, \frac{\sqrt{3}}{3} - 1, \frac{\sqrt{3}}{3} - 1\right)$, ${}^{t}YGY = \frac{2\sqrt{3}}{3}$,

then ${}^{t}XGY = -3$, $X+Y = {}^{t}\left(1, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$,

${}^{t}(X+Y)G(X+Y) = 0$ (*isotropic vector*).

(2) *The triangular counter inequality gives a geometric explanation of the twin − paradox .*

(**6**) ***Classification of Lorentz matrices.***

*We have seen that the set of Lorentz matrices $\mathcal{Lo}$ of order **n** forms a subgroup of $GL_4(\mathbb{R})$, the group of invertible matrices. We can write a Lorentz matrix **M** in the most general writing :*

$$M = exp(\alpha N)\begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix} \text{ where } \alpha \in \mathbb{R},\ N = \begin{bmatrix} 0 & {}^tX \\ X & O \end{bmatrix},\ X \in \mathbb{R}^3$$

$$\text{As } {}^tMGM = G \text{ avec } G = \begin{bmatrix} 1 & {}^t0 \\ 0 & -Id_{\mathbb{R}^3} \end{bmatrix},\ det(M) = \pm 1.$$

But $det(exp(\alpha N)) = e^{Tr(\alpha N)} = e^0 = 1$ *therefore*

$$det(M) = det\left(\begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix}\right) = \varepsilon \cdot det(\Omega).$$

(**2**) *Determination of* ε

Let V_0 *be a non − zero time vector* ${}^tV_0GV_0 > 0$ *and **M** a Lorentz matrix. We have :*

${}^t(MV_0)G(MV_0) = {}^tV_0\,({}^tMGM)\,V_0 = {}^tV_0GV_0 > 0$ *so* MV_0 *is also a time vector.*

Let $\eta = sign\left({}^tV_0G(MV_0)\right)$ *si* $\eta = 1$, *V and* (MV_0) *belong to the same class otherwise* $\eta = -1$, *V and* (MV_0) *belong to different classes.*

*Now let **V** be another non − zero time vector. As previously **MV** is also a time vector.*

As ${}^t(MV)G(MV_0) = {}^tV\,({}^tMGM)\,V_0 = {}^tVGV_0$, *we have the equivalence :*

$Class(V) = Class(V_0) \Leftrightarrow Class(MV) = Class(MV_0)$.

So η *depends only on **M** and is independent of **V**.*

Let us calculate η *for the vector* $E_0 = {}^t(1, 0, 0, 0)$:

$$E_0ME_0 = [1, 0, 0, 0]\begin{bmatrix} \gamma & {}^t[\gamma\vec{\beta}] \\ [\gamma\vec{\beta}] & C \end{bmatrix}\begin{bmatrix} \varepsilon & 0 \\ 0 & \Omega \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \gamma\varepsilon \text{ therefore } \eta = \varepsilon.$$

If $\varepsilon = 1$ *we will say that **M** is orthochronous otherwise if* $\varepsilon = -1$, *we will say that **M** is antichronous.*

As $det(M) = \varepsilon \cdot det(\Omega)$ *we have*

$det(M) = +1 \Leftrightarrow \varepsilon = +1$ *and* $det(\Omega) = +1$ or $\varepsilon = -1$ *and* $det(\Omega) = -1$.

And if $det(M) = -1 \Leftrightarrow \varepsilon = -1$ *et* $det(\Omega) = +1$ *or* $\varepsilon = +1$ *and* $det(\Omega) = -1$.

Let be the following subsets of $\mathcal{Lo}$:

$\mathcal{R}oo = \{M / \varepsilon = +1 \text{ et } det(\Omega) = +1\}$, *the orthochronous rotation group,*

$\mathcal{R}oo$ *is a group called the restricted Lorentz group since for **2** matrices of* $\mathcal{R}oo$:

***M** et **M'**, we have :*

$det(MM'^{-1}) = det(M)det^{-1}(M') = \varepsilon \cdot det(\Omega)\varepsilon' \cdot det(\Omega') = +1$.

$\mathcal{R}oa = \{M / \varepsilon = -1 \text{ et } det(\Omega) = +1\}$, *the **antichronous rotations**,*

$\mathcal{R}eo = \{M / \varepsilon = +1 \text{ et } det(\Omega) = -1\}$, *the **orthochron**ous inversions,*

$\mathcal{R}ea = \{M / \varepsilon = -1 \text{ et } det(\Omega) = -1\}$, *the **antichronous** inversions.*

We note $\mathcal{R}oo \cup \mathcal{R}oa = \{M / det(\Omega) = +1\}$ *is a group, the **rotation group**,*

and $\mathcal{R}oo \cup \mathcal{R}eo = \{M / \varepsilon = +1\}$, *the* ***orthochronous group,***
and $\mathcal{R}oo \cup \mathcal{R}ea = \{M / \varepsilon = +1$ and $det(\Omega) = +1$ or $\varepsilon = -1$ and $det(\Omega) = -1\}$, *the* ***pair group.***

We can also consider the set of matrices of Lorentz such as $\Omega = Id_{\mathbb{R}^3}$, $\varepsilon = +1$ *and* $\vec{\beta} = \beta_x \vec{i}$:

$$\textit{Therefore } M = \begin{bmatrix} \gamma & \gamma\beta_x & 0 & 0 \\ \gamma\beta_x & \gamma & 0 & 0 \\ \multicolumn{2}{c}{\mathbb{T}} & \multicolumn{2}{c}{Id_{\mathbb{R}^2}} \end{bmatrix} \textit{ with } \mathbb{T} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is easy to check that this set is a group of composition : the special group or boost :
We also note that :

$$MM' = \begin{bmatrix} \gamma & \gamma\beta_x & 0 & 0 \\ \gamma\beta_x & \gamma & 0 & 0 \\ \multicolumn{2}{c}{\mathbb{T}} & \multicolumn{2}{c}{Id_{\mathbb{R}^2}} \end{bmatrix} \begin{bmatrix} \gamma' & \gamma'\beta'_x & 0 & 0 \\ \gamma'\beta'_x & \gamma' & 0 & 0 \\ \multicolumn{2}{c}{\mathbb{T}} & \multicolumn{2}{c}{Id_{\mathbb{R}^2}} \end{bmatrix} = \gamma\gamma' \begin{bmatrix} 1 + \beta_x\beta'_x & \beta_x + \beta'_x & 0 & 0 \\ \beta_x + \beta'_x & 1 + \beta_x\beta'_x & 0 & 0 \\ \multicolumn{2}{c}{\mathbb{T}} & \multicolumn{2}{c}{Id_{\mathbb{R}^2}} \end{bmatrix}.$$

We notice that this group is commutative .